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# Advanced Microeconomics / Game Theory

(30-hour lecture in Winter semester 2020/2021)
Thursday 9:45-11:20

**My office hours**: Monday and Thursday, 8:45-9:30

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# Standard microeconomic model of consumer choice

Consumption bundle – a collection of goods $\mathbf{x}=(x_1,\ldots,x_N)$ taken out of a consumption set X: $\mathbf{x}\in X\subset\Re^N$. Normally it is also assumed that $\mathbf{x}\geq\mathbf{0}$.

Let $\mathbf{p}=(p_1,\ldots,p_N)$ be the vector of prices of goods from the consumer's bundle and let m>0 be the consumer's income (money to be spent on the bundle). Then the consumer's budgetary constraint is:

- $p_1x_1+\ldots+p_Nx_N\leq m$, and

- $p_1x_1+...+p_Nx_N=m$

is called <u>budget line</u>. The set

- $\{\mathbf{x}\in X\colon p_1x_1+...+p_Nx_N\leq m\}$

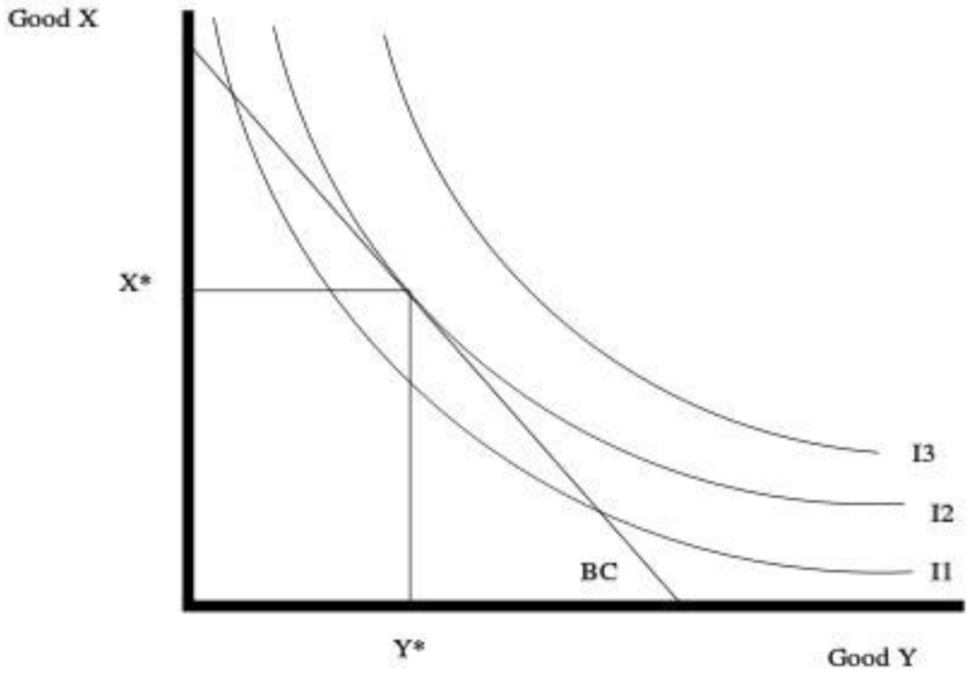
is called <u>budget set</u>.

<u>Utility Maximization Problem</u> (UMP) is solving the problem of consumer's choice under budgetary constraint (with <u>utility</u> as a real function representing preferences)

"Utility", analysed in economics in order to simplify relationships between bundles, was introduced in the 19th century. In mathematical language its definition reads: a <u>utility function</u> $u:X\rightarrow\Re$ representing ≥ is any function such that $\forall \mathbf{x},\mathbf{y}\in X\ [\mathbf{x}\geq\mathbf{y} \Leftrightarrow u(\mathbf{x})\geq u(\mathbf{y})]$.

Expression **x≥y** means bundle **x** is preferred over bundle **y**. Preference relation uses the same symbol as arithmetic relation (greater than or equal to). This, however, shall not lead to any doubts, since it will always be clear from the context whether the formula "x≥y" means "x is preferred over y" or "x is greater than or equal to y". If x and y are consumption alternatives (bundles), i.e. $x,y\in X$, then the formula reads "x is preferred over y", but if x and y are numbers then the formula reads "x is greater than or equal to y".

# Standard model of consumer choice in $\Re^2$



Theorem 1: If $u:X \rightarrow \Re$ is a continuous function, and all prices are positive ($p_i>0$ for $i=1,\ldots,N$) then UMP has a solution $(x_1^*,\ldots,x_N^*)=\mathbf{x^*}(\mathbf{p},m)$

Theorem 2: In the theorem 1 above, if u is strictly concave then the solution is unique

Theorem 3 (Kuhn-Tucker conditions):
In the theorem 1 above, under differentiability assumptions, a solution of the UMP satisfies:

$$\exists\lambda\geq 0\ \forall i=1,\ldots,N\ [\partial u(\mathbf{x}^*)/\partial x_i \leq \lambda p_i\ \&\ x^*_i>0 \Rightarrow \partial u(\mathbf{x}^*)/\partial x_i=\lambda p_i]$$

Theorem 4 (Corollary):
Theorem 3 implies that for a solution of the UMP such that $\forall i=1,...,N$ $[x^*_i>0]$ (internal solution)

$$\partial u(\mathbf{x}^*)/\partial x_i : \partial u(\mathbf{x}^*)/\partial x_j = p_i : p_j$$

for any $i,j=1,...,N$

# Expected utility theory

Bundles are interpreted as lotteries

(D.1) Elements of the set X can be interpreted as lotteries $L=(p_1,\ldots,p_N)$, where $p_1+\ldots+p_N=1$
$\mathcal{L}$ – set of such lotteries; their outcomes – numbered 1,...,N – are predetermined

(T.1) A convex combination of lotteries is also a lottery (with probabilities calculated as convex combinations of probabilities from original lotteries). Easy to prove

<u>(T.2)</u> Preferences ≥ are continuous on $\mathcal{L}$, if for all L,L',L''$\in\mathcal{L}$ the following sets are closed:

$$\{\alpha\in[0,1]\text{: } \alpha L+(1-\alpha)L' \geq L''\}$$

and

$$\{\alpha\in[0,1]\text{: } L'' \geq \alpha L+(1-\alpha)L'\}$$

Easy to prove

<u>Note</u>:

Continuity of preferences is defined in the "standard" consumer theory

<u>(D.2)</u> Preferences ≥ satisfy the <u>independence axiom</u> in $\mathcal{L}$, if for every L,L',L''$\in\mathcal{L}$ and every number $\alpha\in(0,1)$ L≥L' is satisfied if and only if

$$\alpha L+(1-\alpha)L'' \geq \alpha L'+(1-\alpha)L''$$

<u>(D.3) The von Neumann-Morgenstern (vNM) expected utility function</u>, U:

U: has the "expected utility form", when

$\exists\ u_1,\ldots,u_N \in \Re\ \forall\ L=(p_1,\ldots,p_N) \in \mathcal{L}\ [U(L) = u_1 p_1+\ldots+u_N p_N]$

## (D.4) Bernoulli utilities

Numbers $u_i$ from D.3 can be interpreted as utilities of "degenerated lotteries" $L^1=(1,0,\ldots,0),\ldots,L^N=(0,\ldots,0,1)$

(T.3) A utility function $U:\mathcal{L}\rightarrow\Re$ has the "expected utility form" if and only if

$$\forall\ K=1,2,\ldots\ \forall\ L_1,\ldots,L_K\in\mathcal{L}\ \forall\ \alpha_1,\ldots,\alpha_K>0\ [\alpha_1+\ldots+\alpha_K=1\Rightarrow U(\alpha_1L_1+\ldots+\alpha_K L_K) = \alpha_1U(L_1)+\ldots+\alpha_KU(L_K)]$$

<u>Proof</u> $\Leftarrow$

Let $L=(p_1,...,p_N)$. We define degenerated lotteries $L^1,...,L^N$ such that $L^i=(0,...,0,1,0,...,0)$; the *i*th probability is equal to 1. Then $L=p_1L^1+...+p_NL^N$ and $U(L) = U(p_1L^1+...+p_NL^N) = p_1U(L^1)+...+p_NU(L^N) =$ $= p_1u_1+...+p_Nu_N$, where the second to the last equality holds by the assumption.

Proof $\Rightarrow$

Let us consider a combination of lotteries $(L_1,...,L_k;\alpha_1,...,\alpha_k)$, where $L_k=(p_1^k,...,p_N^k)$. Let $L'=\alpha_1 L_1+...+\alpha_k L_k$. Hence it can be calculated that:

$U(L') = U(\alpha_1 L_1+...+\alpha_k L_k) =$

$u_1(\alpha_1 p_1^1+...+\alpha_k p_1^k)+...+u_N(\alpha_1 p_N^1+...+\alpha_k p_N^k) =$

$= \alpha_1(u_1 p_1^1+...+u_N p_N^1)+...+\alpha_k(u_1 p_1^k+...+u_N p_N^k) =$

$\alpha_1 U(L_1)+...+\alpha_k U(L_k)$, where the second to the last equality follows from the assumption.

## (T.4) The expected utility theorem

If a preference relation $\geq$ in $\mathcal{L}$ is rational (i.e. complete and transitive), and satisfies independence and continuity axioms then it can be represented by a vNM function, i.e. numbers $u_n$ can be attributed to each outcome n=1,...N such that:

$$\forall\ L=(p_1,\ldots,p_N), L'=(p'_1,\ldots,p'_N) \in \mathcal{L}$$
$$[L \geq L' \Leftrightarrow u_1p_1+\ldots+u_Np_N \geq u_1p'_1+\ldots+u_Np'_N]$$

Difficult to prove

## (D.5) Lottery with monetary payoffs

The lottery has monetary payments $x_1,\ldots,x_N$, and Bernoulli utilities are a function $u\colon \Re \rightarrow \Re$ of these payments: $u(x_1),\ldots,u(x_N)$.

Warning I (*the Allais paradox*)

There are three outcomes of lotteries:

- $x_1=0$,
- $x_2=1$ million USD, and
- $x_3=5$ million USD

Experiment 1 (most people prefer $L_1$):
Choose between two lotteries:
$L_1=(0,1,0)$ and $L_2=(0.01,0.89,0.1)$

Experiment 2 (most people prefer $L_4$):
Choose between two lotteries:
$L_3=(0.89,0.11,0)$ and $L_4=(0.9,0,0.1)$

Theorem
People who choose $L_1$ in the first experiment and $L_4$ in the second one do not comply with the vNM theory.

<u>Proof:</u>

If the vNM theory was followed, then Bernoulli's utilities would have been applied:

- $u(x_1)=u_1$,
- $u(x_2)=u_2$, and
- $u(x_3)=u_3$.

Proof (cont.):

If the vNM theory was followed, then Bernoulli's

The outcome of the first experiment implies that:

$u_2>0.01u_1+0.89u_2+0.1u_3$.

The outcome of the second one implies that:

$0.9u_1+0.1u_3>0.89u_1+0.11u_2$.

These two inequalities contradict each other, since the second one can rewritten as:

$0.01u_1+0.1u_3>0.11u_2$, and consequently

$0.01u_1+0.1u_3>u_2-0.89u_2$, or

$0.01u_1+0.89u_2+0.1u_3>u_2$

which contradicts the first one.

# Warning II (*the Machina's paradox*)

There are three outcomes of lotteries:

- $x_1$=0,
- $x_2$=10 USD, and
- $x_3$=10,000 USD

It is obvious that $u(x_1)<u(x_2)<u(x_3)$; hence $L_1 \leq L_2 \leq L_3$, where $L_1=(1,0,0)$, $L_2=(0,1,0)$, and $L_3=(0,0,1)$. Thus, by the independence axiom, the lottery $0.001L_2+0.999L_3$ should be preferred over $0.001L_1+0.999L_3$. And yet experiments show that most people choose otherwise.

<u>Conclusion</u>: The expected utility theory may be insufficient to model people's behaviour in some applications

# Questions:

Q-1 The concept of rational preferences over lotteries differs from the "standard" (deterministic) approach to consumer theory by assuming that

[a] preferences may be not transitive

<u>[b] consumers are not certain what outcomes their purchases will imply</u>

[c] consumers do not understand probabilities

[d] preferences may be not complete

[e] none of the above

# Exercises:

E-1 Check whether a utility function U: $\mathcal{L} \to \Re$ has the "expected utility form" if and only if

$$\forall\ L_1, L_2 \in \mathcal{L}\ [U(L_1/2+L_2/2) = U(L_1)/2+U(L_2)/2]$$

# Risk aversion

## (D.6) Risk aversion and risk neutrality implied by Bernoulli utilities

$$\forall L=(p_1,\ldots,p_N)\in\mathcal{L}$$

$$[u(x_1)p_1+\ldots+u(x_N)p_N \leq u(x_1p_1+\ldots+x_Np_N)] \text{ (aversion)}$$

$$\forall L=(p_1,\ldots,p_N)\in\mathcal{L}$$

$$[u(x_1)p_1+\ldots+u(x_N)p_N = u(x_1p_1+\ldots+x_Np_N)] \text{ (neutrality)}$$

<u>(D.7) Certainty equivalent</u> of the lottery $L=(p_1,\ldots,p_N)$ is a number $c(L,u)$ such that

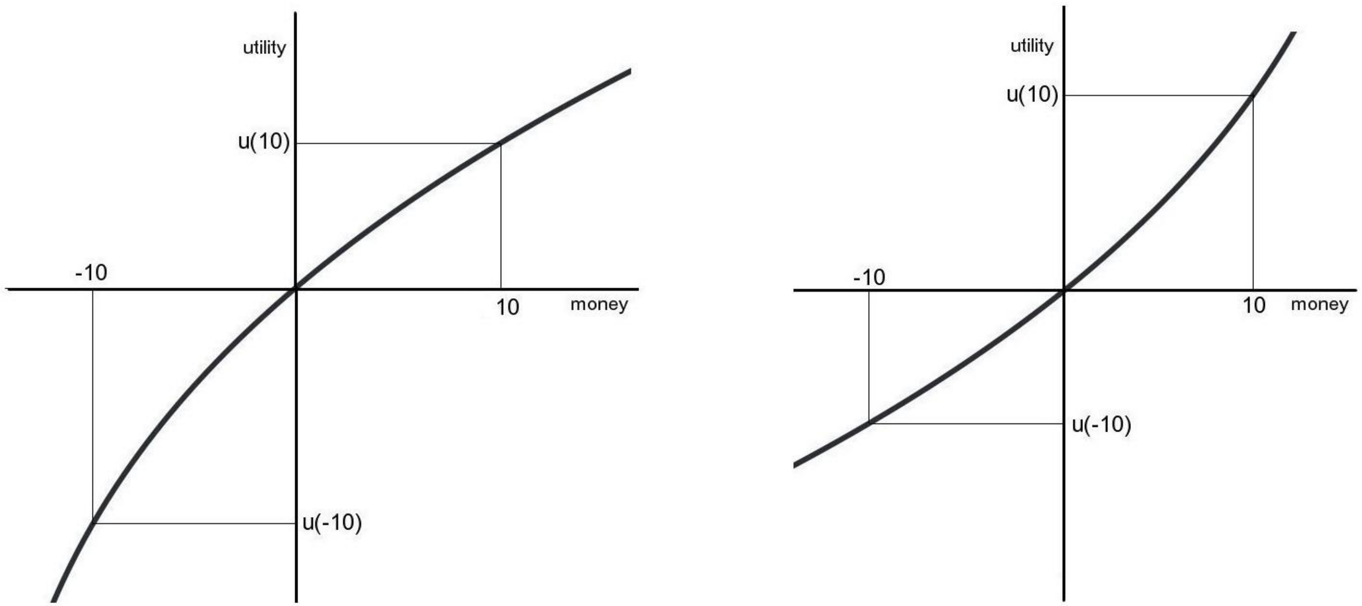
$$u(c(L,u))=u(x_1)p_1+\ldots+u(x_N)p_N$$

<u>(T.5)</u> The following conditions are equivalent:

1. A consumer is risk averse
2. Function u is concave
3. $\forall L=(p_1,\ldots,p_N)\in\mathcal{L}\ [c(L,u) \leq x_1p_1+\ldots+x_Np_N]$

Easy to prove

# Risk averse (left) and risk loving (right)



## (D.8) Arrow-Pratt coefficient of absolute risk aversion

$$r_A(x,u)=-u''(x)/u'(x)$$

assuming that u(.) is a twice-differentiable Bernoulli utility function of money

## Note

Assuming that $u'>0$ and $u''<0$, $r_A$ is a positive number

# (T.6) Comparisons across individuals

Let $u_1$, $u_2$ be Bernoulli utility functions characterizing two individuals. Then the following conditions are equivalent:

1. $r_A(x,u_2) \geq r_A(x,u_1)$ for every $x \in X$
2. There exists an increasing concave function $\psi: \Re \to \Re$ such that $\psi(u_1(x)) = u_2(x)$ for every $x \in X$
3. $c(L,u_2) \leq c(L,u_1)$ for every $L \in \mathcal{L}$

Proof:

There will be only proof of (1)↔(2). The proof of (2)↔(3) is more difficult. It will be assumed that $u_1', u_2' > 0$ (otherwise Arrow-Pratt coefficients cannot be defined).

Proof (cont.)

The most tricky step is to observe that there exists an increasing function $\psi:\Re\to\Re$ such that $\psi(u_1(x))=u_2(x)$ for every $x\in X$ (in fact, there exists an increasing function $\psi:\Re\to\Re$ such that $\psi(u_2(x))=u_1(x)$ for every $x\in X$, but we will use the first statement only). The existence of such a function results from the fact that $u_2$ and $u_1$ are increasing functions (as utilities). As both $u_2$ and $u_1$ are differentiable, it can be assumed that $\psi$ is differentiable too.

Proof (cont.)

Let us differentiate $\psi(u_1(x))=u_2(x)$, yielding $u_2'=\psi'(u_1)u_1'$; and then again, yielding $u_2''=\psi''(u_1)u_1'+\psi'(u_1)u_1''$. Dividing $u_2''$ into $u_2'$ we get (after cancellations): $u_2''/u_2'=\psi''(u_1)/\psi'(u_1)+u_1''/u_1'$. In other words, $r_A(x,u_2)=-\psi''(u_1)/\psi'(u_1)+r_A(x,u_1)$. Thus we will have $r_A(x,u_2)\geq r_A(x,u_1)$ if and only if $-\psi''(u_1)/\psi'(u_1)\geq 0$, i.e. if $\psi$ is concave.

(D.9) Arrow-Pratt coefficient of relative risk aversion

$$r_R(x,u)=-xu''(x)/u'(x)$$

assuming that u(.) is a twice-differentiable Bernoulli utility function of money

Note: $r_R(x,u)=xr_A(x,u)$, and $r_A(x,u)=r_R(x,u)/x$

Corollary: A risk averse individual with fixed (constant with respect to x) coefficient of absolute risk aversion will reveal an increasing (with respect to x) coefficient of relative risk aversion

(D.10) A fair price of a lottery ticket:

The price $t(L)$ equal to the expected payoff, i.e. if $L=(p_1,\dots,p_N)$, and the corresponding payoffs are $x_1,\dots,x_N$, then $t(L)= x_1p_1+\dots+x_Np_N$

Corollary:

For a risk neutral consumer $c(L,u)=t(L)$

# Questions:

Q-2 A risk-averse consumer will

[a] always prefer a lottery with higher payoffs

[b] require that certainty equivalent is higher than the price of the ticket

<u>[c] not buy a lottery ticket whose price is even slightly higher than the expected payoff</u>

[d] always buy a lottery ticket at a fair price

[e] none of the above

# Exercises:

E-2 Prove that the only utility function u implying a constant Arrow-Pratt coefficient of absolute risk aversion $A>0$ is $u(x)=B-Ce^{-Ax}$ for some real number B, and for some constant $C>0$.

# Return-Risk Comparisons

Ordering convention:

$$\{x_1,\ldots,x_N\}=X, \text{ and } x_1<\ldots<x_N$$

## (D.11) First-Order Stochastic Dominance

Let $L=(p_1,\ldots,p_N)$ ,$L'=(p'_1,\ldots,p'_N) \in \mathcal{L}$. L first-order stochastically dominates L' if and only if, for every nondecreasing function $u:X\rightarrow\Re$:

$$p_1u(x_1)+\ldots+p_Nu(x_N) \geq p'_1u(x_1)+\ldots+p'_Nu(x_N)$$

## Example I

Let there be 6 outcomes of throwing dice:

$$x_1=1,\ x_2=2,\ x_3=3,\ x_4=4,\ x_5=5,\ x_6=6.$$

Let L be a lottery that we are used to, i.e.

$$L=(1/6,\ 1/6,\ 1/6,\ 1/6,\ 1/6,\ 1/6).$$

Its average outcome is $E(x) = 3.5$

We will define alternative lotteries L' (with the same outcomes), and analyse how they are perceived by consumers with different attitudes towards risk (demonstrated by different utility functions u).

Example I (cont.)
Examples of alternative utility functions (corresponding to alternative attitudes towards risk; please note that they are nondecreasing, i.e. they satisfy D.11):

- $u(x) = 0 = \text{const}$ (a consumer ignoring money)
- $u(x) = x$ (a risk-neutral consumer)
- $u(x) = x^2$ (a risk-loving consumer)
- $u(x) = x^{1/2}$ (a risk-averse consumer)

Example I (cont.)

The values of the left-hand-side (LHS) of the inequality in D.11 therefore are:

- (0+0+0+0+0+0)/6 = 0
- (1+2+3+4+5+6)/6 = 3.5
- (1+4+9+16+25+36)/6 = 91/6 = 15.17
- (1+1.41+1.73+2+2.24+2.45)/6 = 1.81

Example I (cont.)

Now we define 3 alternative lotteries L':

- L' = (1, 0, 0, 0, 0, 0) with E(x) = 1
- L' = (0, 0, 0, 0, 0, 1) with E(x) = 6
- L' = (0, 0, ½, ½, 0, 0) with E(x) = 3.5

The first and the second are degenerated ones.

For each of the three lotteries, we will calculate the value of the right-hand-side (RHS) of the inequality (in D.11) for the utility functions defined earlier.

# Example I (cont.)

For the lottery L' = (1, 0, 0, 0, 0, 0):

- $u(x)=0=\text{const} \Rightarrow \text{RHS} = 0$
- $u(x)=x \Rightarrow \text{RHS} = 1$
- $u(x)=x^2 \Rightarrow \text{RHS} = 1$
- $u(x)=x^{1/2} \Rightarrow \text{RHS} = 1$

# Example I (cont.)

For the lottery L' = (0, 0, 0, 0, 0, 1):

- $u(x)=0=\text{const} \Rightarrow \text{RHS} = 0$
- $u(x)=x \Rightarrow \text{RHS} = 6$
- $u(x)=x^2 \Rightarrow \text{RHS} = 36$
- $u(x)=x^{1/2} \Rightarrow \text{RHS} = 2.45$

## Example I (cont.)

For the lottery L' = (0, 0, ½, ½, 0, 0):

- $u(x)=0=\text{const} \Rightarrow \text{RHS} = 0$
- $u(x)=x \Rightarrow \text{RHS} = 3.5$
- $u(x)=x^2 \Rightarrow \text{RHS} = 12.5$
- $u(x)=x^{1/2} \Rightarrow \text{RHS} = 3.73$

Example I (cont.)

Summing up these calculations, we can see that
For the lotteries

- L=(1/6,1/6,1/6,1/6,1/6,1/6) and L'=(1, 0, 0, 0, 0, 0): LHS ≥ RHS for all four utility functions analysed
- L=(1/6,1/6,1/6,1/6,1/6,1/6) and L'=(0, 0, 0, 0, 0, 1): the inequality does not hold for utility functions #2, #3, and #4
- L=(1/6,1/6,1/6,1/6,1/6,1/6) and L'=(0, 0, ½, ½, 0, 0): the inequality does not hold for utility function #4

(T.7)

L first-order stochastically dominates L' if and only if, $p_1 \leq p'_1$, $p_1+p_2 \leq p'_1+p'_2, \ldots,$ $p_1+\ldots+p_{N-1} \leq p'_1+\ldots+p'_{N-1}$.

Proof $\Rightarrow$ only:

From the definition of probabilities we obviously know that $p_1+\ldots+p_N=p'_1+\ldots+p'_N=1$. Thus, for any $i=2,\ldots,N-1$ we have $p_1+\ldots+p_{i-1}=1-(p_i+\ldots+p_N)$ and $p_1+\ldots+p_{N-1}=1-p_N$, and $p'_1+\ldots+p'_{i-1}=1-(p'_i+\ldots+p'_N)$ and $p'_1+\ldots+p'_{N-1}=1-p'_N$. This makes the proof easier. The argument will be based on the fact that any implication $p \Rightarrow q$ is equivalent to $\neg q \Rightarrow \neg p$.

Let us assume that $p_1+...+p_{i-1}>p'_1+...+p'_{i-1}$ (for some *i*), i.e. $p_i+...+p_N<p'_i+...+p'_N$. A function $u:X\rightarrow\Re$ defined as $u(x)=0$ for $x\leq x_{i-1}$ and $u(x)=1$ for $x\geq x_i$ is non-decreasing. At the same time $p_1u(x_1)+...+p_Nu(x_N) < p'_1u(x_1)+...+p'_Nu(x_N)$ (because $p_1u(x_1)+...+p_{i-1}u(x_{i-1})=p'_1u(x_1)+...+p'_{i-1}u(x_{i-1})=0$) which means that L does not first-order stochastically dominates L'.

<u>Corollary</u>

In the Example I only the first lottery – L'=(1,0,0,0,0,0) – satisfies the inequalities from T.7 (1/6<1, 1/3<1, 1/2<1, 2/3<1, and 5/6<1). Thus L first-order stochastically dominates over L'. For other pairs of lotteries (L and L'), some inequalities do not hold.

## (D.12) Second-Order Stochastic Dominance

For any two lotteries L and L' with the same mean (i.e. when $p_1x_1+...+p_Nx_N=p'_1x_1+...+p'_Kx_K=\mu$) L second-order stochastically dominates L' (L is less risky than L') if for every nondecreasing concave function $u:X\rightarrow\Re$:

$$p_1u(x_1)+...+p_Nu(x_N) \geq p'_1u(x_1)+...+p'_Ku(x_K)$$

## (D.13) Mean-preserving spread

Let L be a lottery with mean μ, and let $L_x''$ be a family of lotteries indexed with x (outcomes of the lottery L) such that means of each lottery $L_x''$ are equal zero. Thus $L=(p_1,\ldots,p_N)$, $p_1x_1+\ldots+p_Nx_N=\mu$, and $L_x''=(p^x_1,\ldots,p^x_K)$, $p^x_1z^x_1+\ldots+p^x_Kz^x_K=0$. Let L' be a compound lottery with $L_x''$ superimposed on L, where its outcomes are x+z with appropriate probabilities. The mean of L' is thus μ. L' is called a <u>mean preserving spread</u> of L.

(T.8) If L and L' have the same mean then the following statements are equivalent:

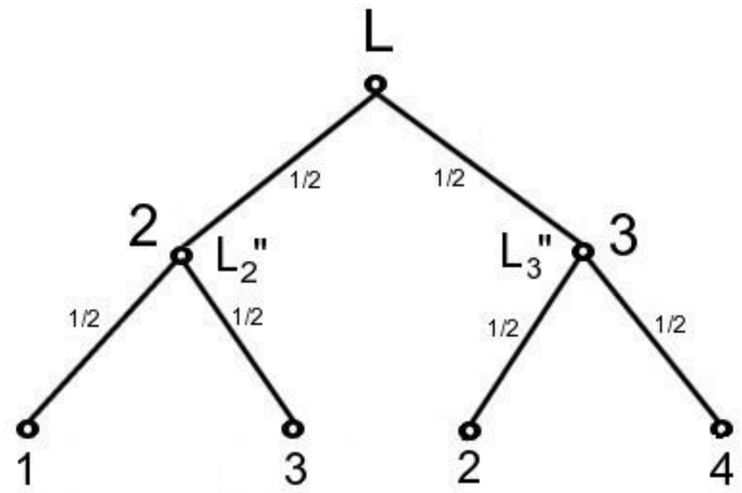
1. L second-order stochastically dominates L'
2. L' is a mean-preserving spread of L

Difficult to prove

Example II

Let $L=(1/2,1/2)$, with $x_1=2$, and $x_2=3$; and let $L_2''=(1/2,1/2)$, with $z^2_1=-1$, $z^2_2=1$, and let $L_3''=(1/2,1/2)$, with $z^3_1=-1$, $z^3_2=1$. Thus we obtain the following compound lottery: $L'=(1/4,1/4,1/4,1/4)$ with outcomes: 1, 2, 3, and 4. As $L_2''$ and $L_3''$ have mean zero, $L'$ is a mean preserving spread of $L$.

# Example II (cont.)



# Questions:

Q-3 A mean preserving spread
[a] lets risk averse players enjoy less risky lotteries
[b] transforms the original lottery into one with a higher variance
[c] yields a lower certainty equivalent than the original lottery
[d] is a lottery with a number of outcomes higher than the original lottery
<u>[e] none of the above</u>

# Exercises:

E-3 Prove that first order stochastic dominance (for lotteries with identical means) implies the second order stochastic dominance, but the converse is not true.

# Introduction to strategic form games

## (D.14) Non-zero sum two-person game

A representation of a decision situation with a table of pairs of numbers ($P_{ij}$,$D_{ij}$). Index $i$=1,...,$m$, where $m$ is the number of strategies (decision variants) for the first player, and $j$=1,...,$n$, where $n$ is the number of strategies (decision variants) for the second player. The numbers $P_{ij}$ are payments to the first player, and $D_{ij}$ – to the second player, if the first chose the $i$th strategy, and the second – the $j$th one. Payments can be interpreted as utilities.

Note: extensions

The D.14 can be generalized into n-person games. The notation becomes more complex, because simultaneous decisions of three or more players cannot be represented by a single matrix. The 'non-zero sum' expression allows for 'zero sum' games as well. 'Non-zero sum' refers to the fact that $P_{ij}+D_{ij}$ is not necessarily zero. If incidentally $P_{ij}+D_{ij}=0$ (i.e. $P_{ij}=-D_{ij}$) the entire framework works, even though certain additional facts can be observed.

<u>(D.15)</u>

The strategy $i_0$ of the first player is called <u>(strictly) dominant</u>, if for any strategy $i$ of the first player, and any strategy $j$ of the second player, $P_{i_0 j} > P_{ij}$; likewise, strategy $i_0$ is <u>(strictly) dominated</u>, if there exists strategy i such that for any strategy $j$, $P_{i_0 j} < P_{ij}$. (analogously for strategies of the second player).

# Example

| | | Second | |
|---|---|---|---|
| | | L | R |
| First | U | (1,3) | (4,1) |
| | D | (0,2) | (3,4) |

U is (strictly) dominant for the First. D is not likely to be played. Therefore the game reduces to:

| | | Second | |
|---|---|---|---|
| | | L | R |
| First | U | (1,3) | (4,1) |

# Example (cont.)

Then R is (strictly) dominated for the Second player, and therefore it is not likely to be played. Thus the game is iteratively reduced to

| | | Second |
|---|---|---|
| | | L |
| First | U | (1,3) |

In other words, the solution is (U,L). Please note that the same result would be obtained if (strictly) dominated strategies of the Second player were eliminated first.

<u>(D.16)</u>
The strategy $i_0$ of the first player is called <u>weakly dominant</u>, if for any strategy $i$ of the first player, and any strategy $j$ of the second player, $P_{i_0 j} \geq P_{ij}$ with strict inequality for at least one strategy of the second player; likewise, strategy $i_0$ is <u>weakly dominated</u>, if there exists strategy i such that for any strategy $j$, $P_{i_0 j} \leq P_{ij}$ with strict inequality for at least one strategy (analogously for strategies of the second player).

# Example

| | | Second | |
|---|---|---|---|
| | | L | R |
| First | U | (5,1) | (4,0) |
| | M | (6,0) | (3,1) |
| | D | (6,4) | (4,4) |

Strategies U and M are weakly dominated by D. If U is eliminated, the game reduces to:

# Example (cont.)

| | | Second | |
|---|---|---|---|
| | | L | R |
| First | M | (6,0) | (3,1) |
| | D | (6,4) | (4,4) |

The second player can eliminate L as weakly dominated by R. The game reduces to:

# Example (cont.)

| | | Second |
|---|---|---|
| | | R |
| First | M | (3,1) |
| | D | (4,4) |

Now the First player can eliminate M, and the game reduces to:

| | | Second |
|---|---|---|
| | | R |
| First | D | (4,4) |

# Example (cont.)

However strategies can be eliminated in a different order. If M is eliminated first, the game reduces to:

| | | Second | |
|---|---|---|---|
| | | L | R |
| First | U | (5,1) | (4,0) |
| | D | (6,4) | (4,4) |

# Example (cont.)

Now R becomes weakly dominated, and the game reduces to:

| | | Second |
|---|---|---|
| | | L |
| First | U | (5,1) |
| | D | (6,4) |

# Example (cont.)

The First player may eliminate U now, and the game reduces to:

| | | Second |
|---|---|---|
| | | L |
| First | D | (6,4) |

Please note that this is a different outcome than in the previous elimination sequence. Hence weak dominance does not justify a reasonable iterative elimination procedure.

(D.17) Nash strategy

Any pair of strategies $(i_0,j_0)$ such that $P_{i_0j_0}=\max_i\{P_{ij_0}\}$ and $D_{i_0j_0}=\max_j\{D_{i_0j}\}$.

(T.9) Corollary of D.15, D.16, and D.17

If players have (either strictly or weakly) dominant strategies, then their pair is a Nash equilibrium

Proof:

If there is $i_0$ such that $P_{i_0j}\geq P_{ij}$ for any j, and there is $j_0$ such that $D_{ij_0}\geq D_{ij}$ for any i, then $(i_0,j_0)$ is a Nash strategy. Please also note that if both inequalities are strict then the equilibrium is unique.

## Example (Prisoner's Dilemma)

| | | Second | |
|---|---|---|---|
| | | C | D |
| First | C | (-12,-12) | (0,-18) |
| | D | (-18,0) | (-1,-1) |

Nash equilibrium is for (C,C) which is the very worst outcome for two players; i.e. Nash equilibrium does not necessarily 'optimize' the global outcome (which – in this case – would be (D,D))

<u>Note</u>
Strategies making a Nash equilibrium do not have to be dominant ones (see e.g. the game from E-4)

<u>(T.10) Alternative definition of Nash equilibrium</u>
If players are in a Nash equilibrium, then – if they wish to maximize their payoffs – neither has a motivation to unilaterally change his or her strategy

<u>Note</u> (behavioural assumption)
Nash equilibrium explains market equilibrium in some circumstances (examples: Cournot and Bertrand models of duopoly)

# Questions:

Q-4 In two-person non-zero sum games Nash equilibrium

[a] guarantees maximum possible payoffs for each of the players

[b] relies on weakly dominant strategies of both players

<u>[c] involves strategies that do not provide incentives for a unilateral change</u>

[d] can always be found iteratively by eliminating strictly dominated strategies

[e] none of the above

# Exercises:

E-4 Provide economic interpretation of a so-called coordination game. Identify its Nash equilibria (if any):

| | Second | | |
|---|---|---|---|
| First | | L | R |
| | U | (1,1) | (-1,-1) |
| | D | (-1,-1) | (1,1) |

# Strategic form games

## Note

There exist games with no Nash equilibrium. See e.g.

| | | M | |
|---|---|---|---|
| | | Y | N |
| F | Y | (4,2) | (-2,3) |
| | N | (2,1) | (-1,0) |

## (D.18) Mixed strategies

Strategies defined so far are called *pure*. A game can be defined where pure strategies are selected by players randomly with certain probabilities $\boldsymbol{\pi}=(\pi_1,...,\pi_m)$ and $\boldsymbol{\delta}=(\delta_1,...,\delta_n)$, respectively (for the first and the second player), where $\pi_1,...,\pi_m \geq 0$, $\pi_1+...+\pi_m=1$ and $\delta_1,...,\delta_n \geq 0$, $\delta_1+...+\delta_n=1$. The pair $(\boldsymbol{\pi},\boldsymbol{\delta})$ is called a mixed strategy selection.

## (D.19) Payoffs in games with mixed strategies

If the players select mixed strategies, then the payoffs are understood as expected payoffs from their pure strategies. In other words, the payoff for the first is $\sum_{ij}\pi_i\delta_j P_{ij}$, and for the second is $\sum_{ij}\pi_i\delta_j D_{ij}$.

## Note

A 'traditional' game (with pure strategies) can be interpreted as a mixed-strategy game where probabilities can be either 0 or 1 (they are 'degenerate')

## Note

Nash equilibrium definition can be generalized for mixed strategies. In other words, a pair of strategies $(\boldsymbol{\pi^0},\boldsymbol{\delta^0})$ is a Nash equilibrium, if

- $\sum_{ij}\pi_i^0\delta_j^0P_{ij}=\max_\pi\{\sum_{ij}\pi_i\delta_j^0P_{ij}\}$, and
- $\sum_{ij}\pi_i^0\delta_j^0D_{ij}=\max_\delta\{\sum_{ij}\pi_i^0\delta_jD_{ij}\}$.

## (T.11)

For every non-zero sum two-person game there exists a Nash equilibrium for mixed strategies (proof can be derived from the Brouwer's fixed-point theorem; difficult).

# Example

The game from the note has a Nash equilibrium in mixed strategies:

If p is the probability of choosing Y for the player F, and q is the probability of choosing Y for the player M, then (1/2,1/2;1/3,2/3) is the Nash equilibrium in mixed strategies (when p=1/2, and q=1/3 the players do not have incentives to unilaterally change these probabilities).

<u>(D.20) Game in a normal (condensed, strategic) form</u>, $\Gamma_N = [I, S_1\times...\times S_I, (u_1,...,u_I)]$, where

1. $S_i$ – set of strategies of player *i* ($s_i \in S_i$)
2. $u_i(s_1,...,s_I)$ – a payoff function whose values can be interpreted as expected utilities (in the von Neumann-Morgenstern sense) of outcomes (perhaps probabilistic ones)

## Note

D.20 generalizes D.14 by letting the number of players be I (instead of 2), introducing abstract "strategies" (instead of "rows" and "columns"), and utilities instead of payoffs.

(D.21) Notational convention: $s_{-i} = (s_1,\ldots,s_{i-1},s_{i+1},\ldots,s_I)$

(D.22) Correlated equilibrium

Probability distribution p over the set of strategies $S=S_1\times\ldots\times S_I$ such that for every player i and every permutation $d_i : S_i \rightarrow S_i$

$$\sum_{s\in S} p(s)u_i(s_i,s_{-i}) \geq \sum_{s\in S} p(s)u_i(d_i(s_i),s_{-i})$$

(T.12)

Nash equilibrium is a correlated equilibrium for a degenerate distribution p. More precisely: $p(i_0,j_0)=1$, and for every $(i,j)\neq (i_0,j_0)$, $p(i,j)=0$ (easy to prove).

# Example I

There are two Nash equilibria in the following so-called coordination game from E-4: (U,L), and (D,R)

| | Second | | |
|---|---|---|---|
| | | L | R |
| First | U | (1,1) | (-1,-1) |
| | D | (-1,-1) | (1,1) |

However, if the players choose their strategies randomly and independently (with probabilities 1/2), their average payoffs will be 0.

Example I (cont.)

If they choose strategies based on a random but publicly observed signal s, such like, say, s=Heads or s=Tails; and if s=Heads then First plays U and Second plays L, while if s=Tails then First plays D and Second plays R. This is a correlated equilibrium with payoffs equal to 1 enjoyed by both players.

# Example II

In the following game

| | Second | |
|---|---|---|
| First | L | R |
| U | (5,1) | (0,0) |
| D | (4,4) | (1,5) |

there are two Nash equilibria (U,L) and (D,R). If the players choose their strategies randomly and independently (with probabilities 1/2), then – on average – they will get 2.5 each.

# Example II (cont.)

Please note that no publicly available signal can motivate them to play (D,L), i.e. the combination yielding the maximum sum of payoffs.

# Example III

Let us consider the following 3-person game:

| | L | R |
|---|---|---|
| U | 0,1,3 | 0,0,0 |
| D | 1,1,1 | 1,0,0 |

A

| | L | R |
|---|---|---|
| U | 2,2,2 | 0,0,0 |
| D | 2,2,0 | 2,2,2 |

B

| | L | R |
|---|---|---|
| U | 0,1,0 | 0,0,0 |
| D | 1,1,0 | 1,0,3 |

C

First chooses U or D, Second chooses L or R, Third chooses the payoff matrix, and the unique Nash equilibrium is (D,L,A) with payoffs 1 to each of the players

# Example III (cont.)

Let us assume that players choose a correlating device such that s=Heads or s=Tails, and the signal is revealed to First and Second, but not to the Third player. A correlated equilibrium is

$$(U/2, D/2, L/2, R/2, 0A, 1B, 0C)$$

with expected payoffs

$$(2,2,2).$$

If s=Heads then First plays U, otherwise D. If s=Heads then Second plays L, otherwise R.

# Example III (cont.)

Please note that if the outcome of flipping the coin was revealed to the Third player, there would be no correlated equilibrium.

## Note

Interpretations of Nash equilibrium

1. Self-enforcing agreement
2. Stable social convention
3. Evolutionary stable behaviour

# Questions:

Q-5 Correlated equilibrium generalizes the concept of Nash equilibrium by

[a] assuming that players may communicate with each other directly

[b] letting some of them know others' decisions in advance

<u>[c] assuming that some players may observe some random signals making others' decisions more likely</u>

[d] allowing strategies to be chosen with certain probabilities rather than deterministically

[e] none of the above

# Exercises:

E-5 Demonstrate that in Example II, by an appropriate design of a signalling device, the players can guarantee expected payoffs 3 1/3 each in equilibrium. However they cannot achieve the best joint outcome (4,4) in equilibrium.

# Bayesian games (games with incomplete information)

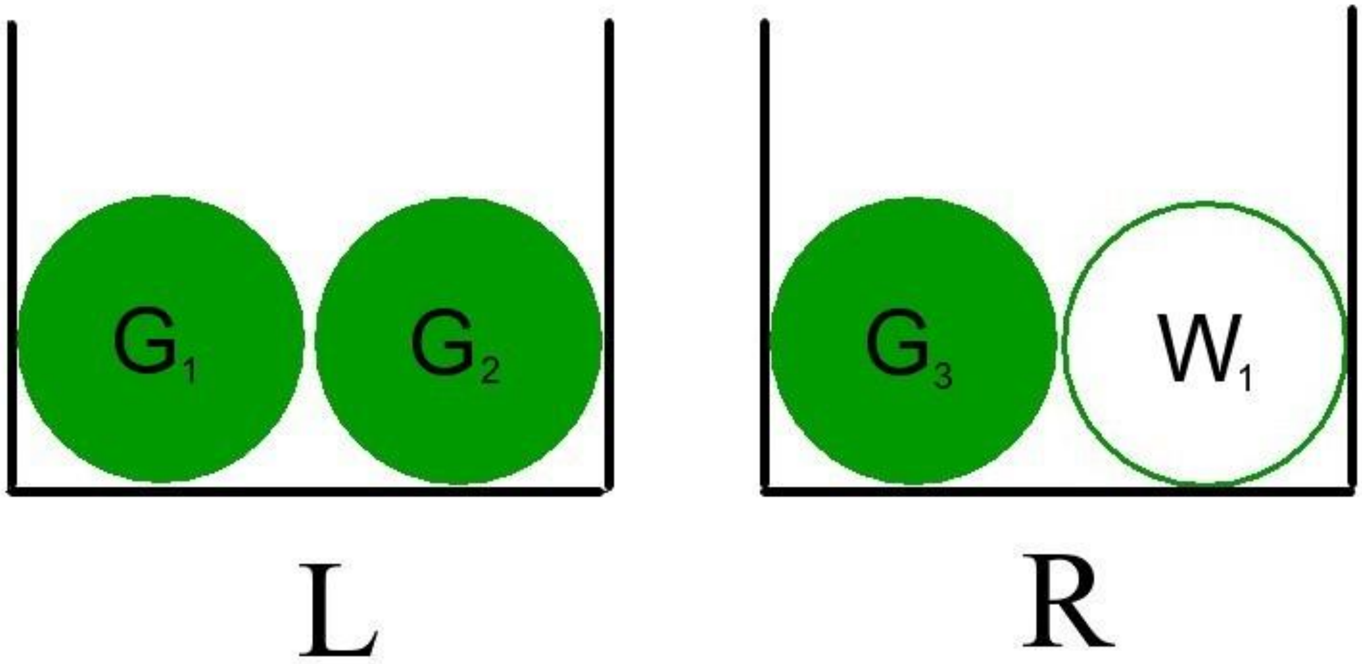
## Conditional probability

$$P(A \mid B) = P(A \cap B)/P(B)$$

## Bayes formula (the simplest version)

$$P(A \mid B) = P(B \mid A) \cdot P(A) \,/\, P(B)$$

# Example I (sampling four balls from two boxes)



## Example I (cont.)

- $P(G|L) = P(G \cap L)/P(L) = (1/2)/(1/2) = 1$,
- $P(G|R) = P(G \cap R)/P(R) = (1/4)/(1/2) = 1/2$,
- $P(W|L) = P(W \cap L)/P(L) = 0/(1/2) = 0$,
- $P(W|R) = P(W \cap R)/P(R) = 1/4/(1/2) = 1/2$.

## Corollary:

*A priori* beliefs may influence decisions

<u>Example II</u> (modified prisoner's dilemma)

Let there be two games

| "I" | | Second | |
|---|---|---|---|
| | | C | D |
| First | C | (-5,-5) | (-1,-10) |
| | D | (-10,-1) | (0,-2) |

and

# Example II (cont.)

| "II" | | Second | |
|---|---|---|---|
| | | C | D |
| First | C | (-5,-11) | (-1,-10) |
| | D | (-10,-7) | (0,-2) |

The difference between "I" and "II" is in payoffs for the Second player in the "C" strategy. They read -11 or -7 instead of -5 or -1 (they are lowered by 6 in both cases).

## Example II (cont.)

In other words, the difference between "I" and "II" is in the type of the Second player. The First player does not know what is the type of the Second player, but has certain beliefs. Namely, the first player believes that the game is of the "I" type with probability $\mu$, or of the type "II" with probability $1-\mu$. The Second player knows his/her type.

# Example II (cont.)

If the game is of the type "I" then the strictly dominant strategy of the Second player is C, but it is D, if the game is of the type "II". Thus the First player – who believes that the game is of the "I" type with probability μ – should choose C if

$$-10\mu+0(1-\mu)<-5\mu-(1-\mu),$$

or – alternatively – choose D if

$$-10\mu+0(1-\mu)>-5\mu-(1-\mu);$$

if the equality holds, the First player is indifferent between choosing C and D.

# Example II (cont.)

Thus the solution (the best strategy) for the First player is C if $\mu>1/6$, and D if $\mu<1/6$.

(D.23)

A Bayesian non-zero sum two-person game is a game where payoffs $P_{ij}(\theta_P)$ and $D_{ij}(\theta_D)$ depend on realization of random variables $\theta_P$ and $\theta_D$. It is assumed that: (1) distributions of $\theta_P$ and $\theta_D$ are known to both players; and (2) realization of $\theta_P$ is known to the First player only, and of $\theta_D$ – to the Second one only.

# Note

In the example it was assumed that $\theta_P$ had a degenerate distribution that is $P_{ij}$ values were known with certainty. At the same time, $\theta_D$ had a two-point distribution such that $D_{CC}$=-11 and $D_{DC}$=-7 with the probability of μ, and $D_{CC}$=-5 and $D_{DC}$=-1 with the probability of 1-μ.

$D_{CD}$=-10, and $D_{DD}$=-2 for any realization of the random variable $\theta_D$.

## Note

Definition D.23 can be generalized so as to apply to n-person games

## (D.24)

In a Bayesian game D.23, a (pure) strategy (decision rule) for the First player is $S_P(\theta_P)$ and for the Second player – $S_D(\theta_D)$ with payoffs

$$E_\theta P_{S_P(\theta_P)S_D(\theta_D)}(\theta_P) \text{ and } E_\theta D_{S_P(\theta_P)S_D(\theta_D)}(\theta_D),$$

where $E_\theta$ is the expected value given the distribution of $\theta_P$ and $\theta_D$, $S_P \in \{1,\ldots,m\}$ and $S_D \in \{1,\ldots,n\}$

<u>(D.25)</u>

In a Bayesian game D.23, a (pure strategy) Bayesian Nash equilibrium is a pair $(S_P(\theta_P), S_D(\theta_D))$ of D.24 decision rules such that for any other strategies $S'_P(\theta_P), S'_D(\theta_D)$:

$$E_\theta P_{S_P(\theta_P) S_D(\theta_D)}(\theta_P) \geq E_\theta P_{S'_P(\theta_P) S_D(\theta_D)}(\theta_P)$$

and

$$E_\theta D_{S_P(\theta_P) S_D(\theta_D)}(\theta_D) \geq E_\theta D_{S_P(\theta_P) S'_D(\theta_D)}(\theta_D)$$

## Note

In the example above the Bayesian Nash equilibrium is a pair of strategies $(S_P(\theta_P), S_D(\theta_D))$ where $S_P(\theta_P)=S_P=$(C if $\mu>1/6$, D if $\mu<1/6$), and $S_D(\theta_D)=$(C if the realization of $\theta_D$ is type "I", D if the realization of $\theta_D$ is type "II"). Both players know that the distribution of $\theta_P$ is degenerate (there is only one realization of $\theta_P$), and the distribution of $\theta_D$ has two values (attained with probabilities $\mu$ and $1-\mu$). The First player does not know the realization of $\theta_D$, but the Second one does.

# Questions:

Q-6 In a game with incomplete information (defined as a Bayesian game)

[a] players do not know their preferences with certainty

[b] players know their own payoffs resulting from strategies applied (subject to others' strategies applied)

[c] players always know the others' payoffs resulting from others' strategies applied

[d] incompleteness of information refers to the fact players cannot perfectly predict payoffs resulting from their strategies

[e] none of the above

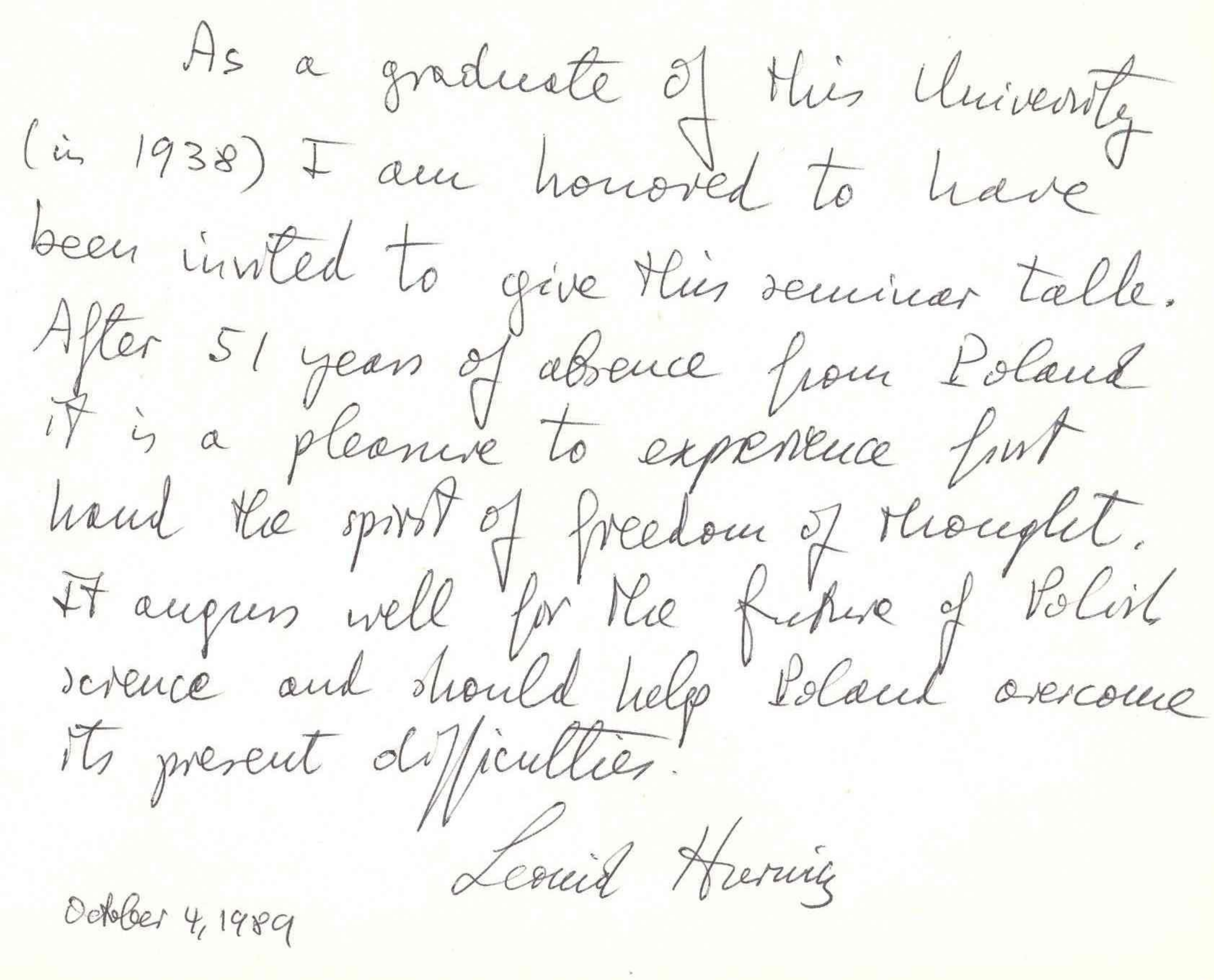
# Exercises:

E-6 Please calculate a Bayesian Nash equilibrium for the game defined like in the example, but with the following payoff matrices:

| "I" | | Second | |
|---|---|---|---|
| | | C | D |
| First | C | (-5,-5) | (-1,-10) |
| | D | (-8,-1) | (0,-2) |

| "II" | | Second | |
|---|---|---|---|
| | | C | D |
| First | C | (-5,-11) | (-1,-10) |
| | D | (-8,-7) | (0,-2) |

# Mechanism design



As a graduate of this University (in 1938) I am honored to have been invited to give this seminar talk. After 51 years of absence from Poland it is a pleasure to experience first hand the spirit of freedom of thought. It augurs well for the future of Polish science and should help Poland overcome its present difficulties.

Leonid Hurwicz

October 4, 1989

Leonid Hurwicz (1917-2008) was awarded Nobel Prize in economics in 2007 for his role in Mechanism Design (among other things)

'Mechanism design' =

Constructing a game which helps to overcome asymmetric information

Asymmetric information

The buyer has less information about the commodity than the seller or *vice versa*; acquiring information is possible, but costly.

*Incentive compatibility* = motivating for efficiency in a *principal-agent* setting

Model

- x – employee's effort
- y=f(x) – product (we assume that its price is equal to 1)
- s(y) or s(x) – employee's salary
- c(x) – cost born by the employee
- $u^0$ – employee's aspiration level: $s(f(x))-c(x) \geq u^0$ (*participation constraint*)

<u>(T.13)</u>

*Incentive compatibility constraint* is:

$s(f(x^*))-c(x^*) \geq s(f(x))-c(x)$ for all x, where

- $x^*$ maximizes $f(x)-s(f(x))$, i.e. $f(x)-c(x)-u^0$ that is (by conventional assumptions):
- $MP(x^*) = MC(x^*)$

Proof:

The *incentive compatibility constraint* means that $x^*$ maximizes the expression $s(f(x))-c(x)$, i.e. the net benefit for the agent. The theorem (first bullet) states that the same $x^*$ maximizes $f(x)-s(f(x))$, i.e. the net benefit for the principal. By the rationality of the principal, $s(f(x))=c(x)+u^0$ (the principal should offer the employee exactly what is expected). The expression $f(x)-c(x)-u^0$ is maximized (under the concavity assumption) for $x^*$ such that $f'(x^*)=c'(x^*)$ (the second bullet).

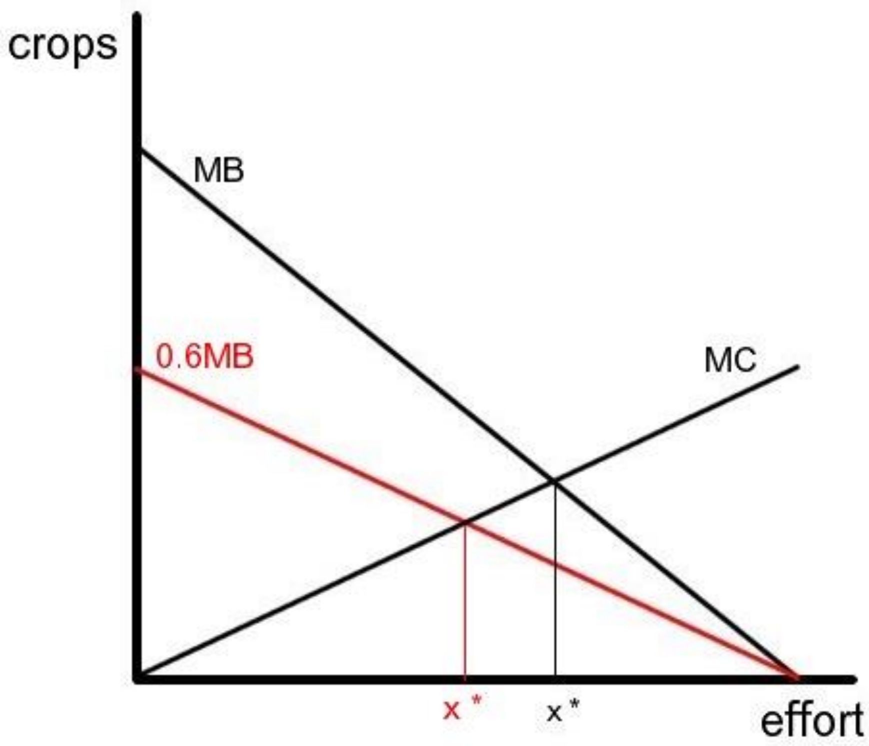
## Corollary

Incentive compatibility constraint is satisfied if the employee is a so-called *residual claimant* (i.e. has the right to get the entire marginal product of his or her effort when it is close to x*)

Examples (of incentive-compatible contracts)

- Rental payment, R: $s(f(x)) = f(x)-R$, where R is derived from the participation constraint: $f(x^*)-c(x^*)-R = u^0$
- Hourly (daily) salary rate *w* plus flat rate *K* so that: $s(x) = wx+K$, where $w=MP(x^*)$, and K is derived from the participation constraint: $wx+K-c(x) = u^0$
- Threshold condition (*take-it-or-leave-it*), payment B, if $x \geq x^*$ (alternatively: if $y \geq f(x^*)$): the amount B is calculated from the participation constraint: $B-c(x^*) = u^0$ (assuming that $B \leq MP(x^*)$); otherwise the agent gets nothing

# <u>Example</u> (of an incentive-incompatible contract)



## Note

The *principal-agent* approach can be extended to situations with many agents. The *incentive compatibility* condition should

- integrate interests of the principal and agents; and
- reveal information about agents' preferences (thus reducing the information asymmetry)

# Example I: Judgement of Solomon

Two women quarrelled about who was the mother of a baby. Each of them knew the truth, but nobody else did. Solomon was to judge. He designed a game. Each woman was to say 'Yes' or 'No' to his proposal to cut the baby in two. The payoff matrix in this game would be:

| | | Second | |
|---|---|---|---|
| | | Y | N |
| First | Y | (-100,1) | (-100,0) |
| | N | (0,1) | (0,0) |

## Example I (cont.):

Solomon assumed that the real mother would say N, and the fake one would say Y. Based on the outcome (N,Y) he identified the First as the real mother thus solving his objective which was to make a fair judgement. The trick can be considered a successful implementation of 'mechanism design'. Of course, if the game were to be played repeatedly, its 'payoffs' would probably change.

# Examples II: auctions

The principal has an object for sale. It can be purchased by either of two agents which value the object $v_1$ and $v_2$, respectively. Each valuation is known only to the agent who has it. Nevertheless the principal and both agents know the distributions of both values treated as random variables: $\theta_1$ for the first valuation and $\theta_2$ – for the second. Agents are supposed to indicate their valuations as sealed bids $s_1$ and $s_2$ (which do not have to coincide with $v_1$ and $v_2$). The object goes to the agent who offered the higher bid.

If both bidders offer the same bid then the winner is selected randomly with probabilities 1/2 and 1/2. Auctions vary with respect to what is the amount paid by the winner.

1. First-Price Auction
   The winner pays the amount from his/her own bid

2. Second-Price Auction
   The winner pays the amount from the loser's bid

Types of auctions:

- 'Sealed bid' (our examples)
- Descending
- Ascending

Types vary in terms of information revealed (e.g. ascending reveals more information than descending)

(T.14)

Let us assume that $\theta_1$ and $\theta_2$ have the same distributions. In both variants of auctions the truthful revelation of preferences makes a Nash equilibrium in the 'bidding game' (the payoff of the loser is 0, and the payoff of the winner is $v_i$-$s_i$ in the first-price auction or $v_i$-$s_{-i}$ in the second-price auction)

Proof:

Let us consider the first-price auction first. The winner pays $s_i$, and gets the net benefit $v_i$-$s_i$. If $s_i$>$v_i$ then the net benefit would be negative and the winner would have a motivation to be the loser rather than the winner. If $s_i$<$v_i$ then the winner risks loosing the auction and hence – $s_i$ makes the preferred bid. Now let us consider the second-price auction. The motivation for the winner not to overstate the bid is even higher than before. A likely motivation to understate in order to pay less disappears, because the winner pays the bid of loser.

(T.15) (Revenue Equivalence Theorem)

In both types of auctions the expected revenue of the principal (who organizes an auction) is the same.

Proof:

The seller receives $s_i$ of the winner (in the first-price auction) or $s_i$ of the loser (in the second-price auction). By T.14, bidders indicate their true valuations (i.e. $s_i=v_i$, for $i=1,2$). As $v_i$ are sampled from the same distributions, then $Ev_1=Ev_2$.

<u>(T.16)</u> (Inefficiency Theorem)

Let us assume that $\theta_1$ and $\theta_2$ have strictly positive distributions on intervals that overlap at least partially. Neither of the auctions can – in general – provide an outcome that satisfies three conditions:

- individual rationality,
- incentive compatibility, and
- budget balance (any payoffs for the principal need to be financed from agents' payments)

(difficult to prove)

# Questions:

Q-7 In the second-price auction with more than two agents
[a] there are at least two winners
[b] the winner always pays less than what he or she stated in the sealed bid
[c] the losers have incentives to overstate their valuations
[d] on average, the principal receives less than in the first-price auction
<u>[e] none of the above</u>

# Exercises:

E-7 A craftsman leases a machine from an entrepreneur for a fixed rental payment R. The craftsman sells products of the machine in a competitive market at the price of 1. Production depends on the amount of labour L devoted to the production process according to the formula $f(L) = 40\ L^{1/2}$. The cost of labour is 0.5. The craftsman would not accept a lease unless his net revenues from the sales (i.e. after the cost of labour and rent have been paid) is at least 300. What a rent R should be charged by the entrepreneur if the contract was to be accepted and incentive compatible?

# Players do not move simultaneously, and they remember all their previous moves

(D.26) Extensive Form Game

$\Gamma_E = [\mathcal{X}, \mathcal{A}, n, p, \alpha, \mathcal{H}, H, \iota, \rho, u]$, where:

1. A finite set of nodes $\mathcal{X}$, a finite set of possible actions $\mathcal{A}$, and a finite set of players $\{1,\ldots,n\}$

2. A function p: $\mathcal{X} \rightarrow \mathcal{X} \cup \{\varnothing\}$ specifying a single immediate <u>predecessor</u> of each node x; for every $x \in \mathcal{X}$ p(x) is non-empty except for one, designated as the <u>initial node</u>, $x_0$. The immediate successor nodes of x are thus $s(x)=p^{-1}(x)$. All predecessors and all successors of x can be found by iterating operations p and s. It is assumed that for any x and for any k=1,2,... $p(x) \cap s^k(x)=\varnothing$ (if $s^k(x)$ is defined). The set of <u>terminal nodes</u> $T=\{x \in \mathcal{X}: s(x)=\varnothing\}$. All other nodes ($\mathcal{X} \setminus T$) are called <u>decision nodes</u>.

3. A function $\alpha: \mathcal{X}\backslash\{x_0\} \rightarrow \mathcal{A}$ giving the action that leads to any non-initial node x from its immediate predecessor p(x) and satisfying the condition that:

   if $x', x'' \in s(x)$ and $x' \neq x''$, then $\alpha(x') \neq \alpha(x'')$.

   The set of <u>choices</u> available at decision node x is

   $c(x) = \{a \in \mathcal{A}: a = \alpha(x') \text{ for some } x' \in s(x)\}$

4. A <u>family of information sets</u> $\mathcal{H}$ and a function H:$\mathcal{X} \rightarrow \mathcal{H}$ assigning each decision node to an <u>information set</u> H(x)$\in\mathcal{H}$. Thus, the information sets in $\mathcal{H}$ form a partition of $\mathcal{X}$. It is required that all decision nodes assigned to the same information set have the same choices available, i.e.:

$$H(x)=H(x') \Rightarrow c(x)=c(x').$$

Hence choices available at information set H can be defined as

$$C(H) = \{a\in\mathcal{A}: a\in c(x) \text{ for } x\in H\}$$

5. A function $\imath:\mathcal{H} \rightarrow \{0,1,...,n\}$ assigning each information set in $\mathcal{H}$ to a player (or to nature formally identified as the player 0), who moves at the decision nodes in that set. We can define the <u>family of player's *i* information sets</u> as

$$\mathcal{H}_i=\{H\in\mathcal{H}: i=\imath(H)\}$$

6. A function $\rho: \mathcal{H}_0\times\mathcal{A} \rightarrow [0,1]$ assigning probabilities to actions at information sets where <u>nature moves</u> and satisfying $\rho(H,a)=0$ if $a\notin C(H)$, and $\sum_{a\in C(H)}\rho(H,a)=1$ for all $H\in\mathcal{H}_0$

7. A family of <u>payoff functions</u> $u=(u_1,...,u_n)$ assigning utilities to the players for each terminal node that can be reached, $u_i:T \rightarrow \Re$. In order to be consistent with the theory of expected utility, values of the functions $u_i$ should be interpreted as Bernoulli utilities.

# Example: Tic Tac Toe (classic)

| | | |
|---|---|---|
| X | | |
| | | |
| | | |

| | | |
|---|---|---|
| X | O | |
| | | |
| | | |

| | | |
|---|---|---|
| X | O | |
| X | | |
| | | |

| | | |
|---|---|---|
| X | O | |
| X | | |
| O | | |

| | | |
|---|---|---|
| X | O | X |
| X | | |
| O | | |

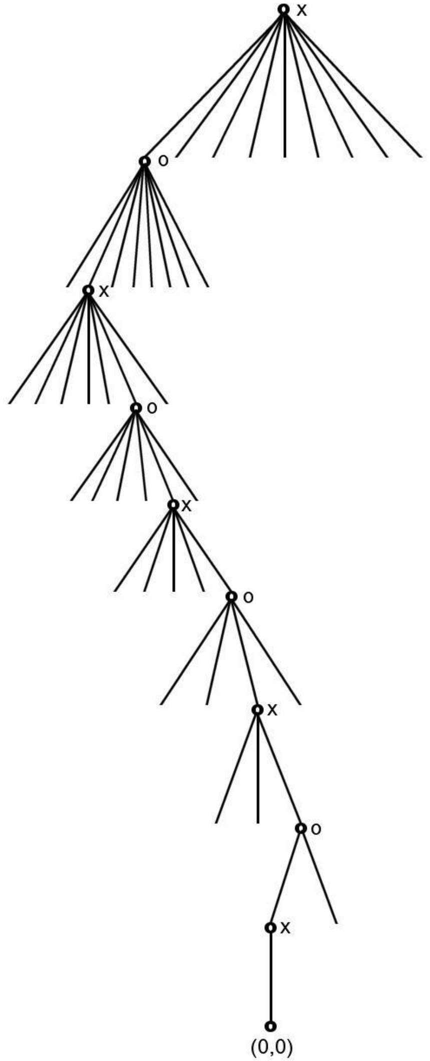
| | | |
|---|---|---|
| X | O | X |
| X | | |
| O | | O |

| | | |
|---|---|---|
| X | O | X |
| X | | |
| O | X | O |

| | | |
|---|---|---|
| X | O | X |
| X | O | |
| O | X | O |

| | | |
|---|---|---|
| X | O | X |
| X | O | X |
| O | X | O |

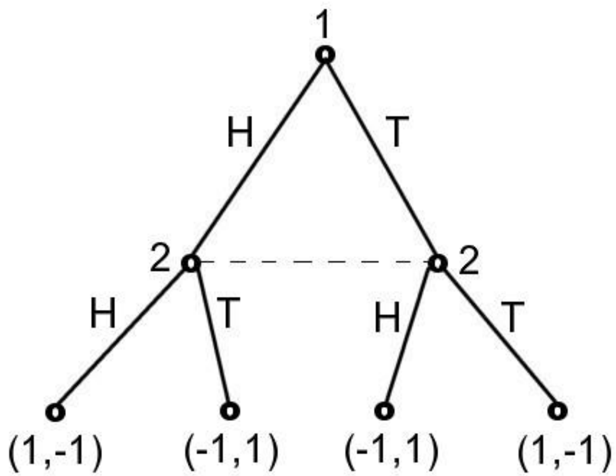
# Example: Tic Tac Toe (mathematical)



# Example

## Matching pennies

One player chooses Heads or Tails. The other one independently (perhaps simultaneously) chooses Heads or Tails as well. They disclose their choices.



# Example (cont.)

In case there is HH or TT, the second pays 1 to the first; otherwise the first pays 1 to the second. Independence (perhaps simultaneity) of both choices is reflected by the fact that the nodes where the second player makes the choice belong to the same information set. In the diagram ('game tree') this is reflected by the dotted line.

<u>(D.27)</u>

Let $\mathcal{H}_i$ denote the family of information sets of the player *i*, $\mathcal{A}$ – set of possible actions, and $C(H)\subset\mathcal{A}$ – the set of actions available at information set H. A <u>strategy</u> for the player *i* is a function $s_i$: $\mathcal{H}_i \rightarrow \mathcal{A}$ such that $s_i(H)\in C(H)$ for all $H\in\mathcal{H}_i$

(T.17)

For every extensive form game (D.26) there is a unique normal form game (D.20), but not *vice versa* (i.e. the same normal form game may correspond to several extensive form games)

Proof:

The payoff matrix is defined by payments taken from the terminal nodes. 'Paths' leading to the terminal nodes define strategies of the players. The not '*vice versa*' part is implied by the fact that the same outcome can be reached by applying the same strategies although in a different sequence.

## Example

The 'Matching pennies' game depicted in the tree above corresponds to the following payoff matrix:

| | | Second | |
|---|---|---|---|
| | | H | T |
| First | H | (1,-1) | (-1,1) |
| | T | (-1,1) | (1,-1) |

(D.28) Sequential rationality principle

Each player's strategy should contain actions that are optimal for every node (taking into account other players' strategies)

(D.29) Finite game with perfect information

Every information set contains only one node and the number of nodes is finite

## (T.18) Backward induction

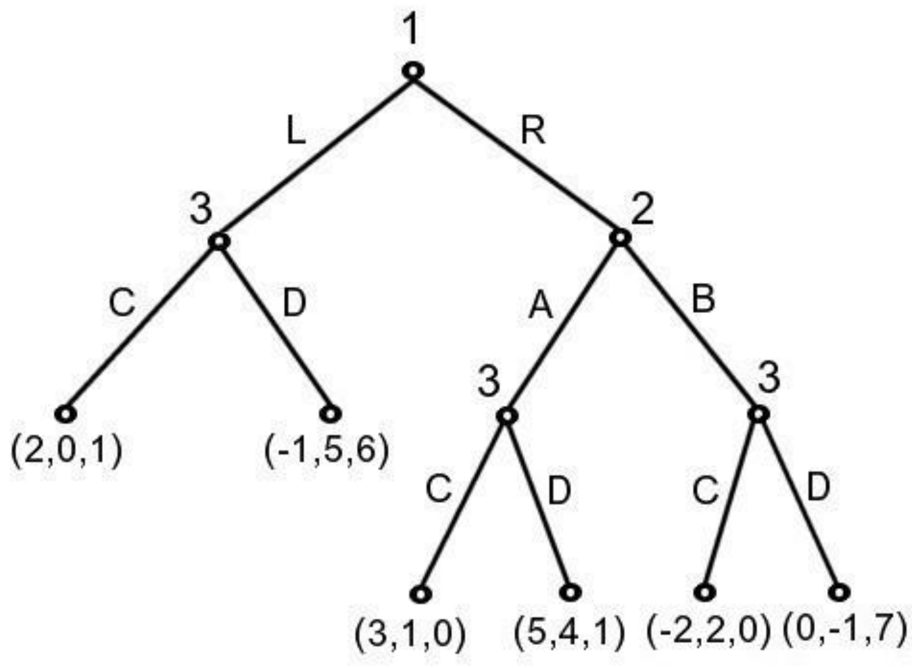
Sequential rationality principle is satisfied if an optimum action for p(x) is determined once an optimum action for x is determined (i.e. anticipating an optimum solution at x)

Proof:

For finite games with perfect information backward induction boils down to determining outcomes for each terminal nodes x, determining optimum actions for preceding nodes p(x), assigning them payoffs that result from these optimum actions, and eliminating remaining strategies. This procedure is then iterated for earlier nodes p(p(x)) and so on, until all the nodes are exhausted.

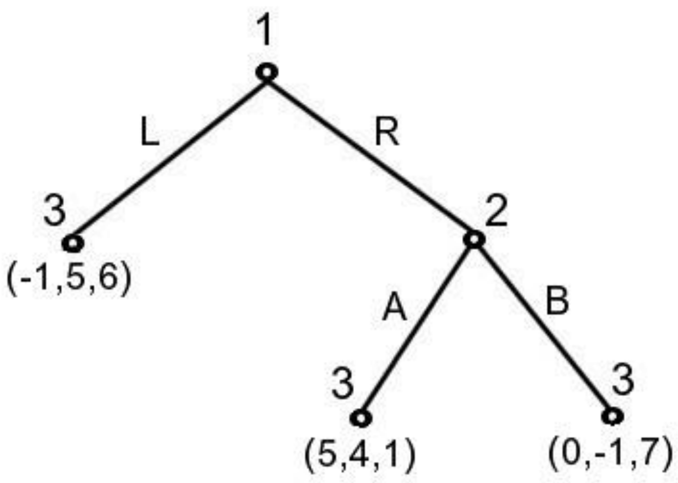
# Example

The extensive form game



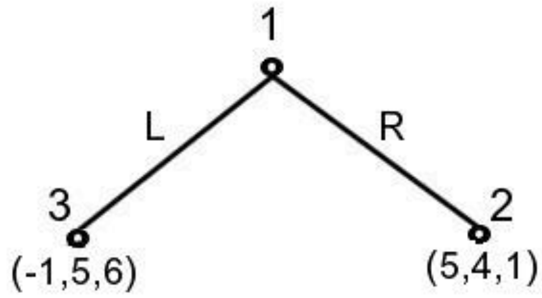
# Example (cont.)

By applying the sequential rationality principle the game can be reduced to



# Example (cont.)

By applying it once again, the game reduces further to




By looking at the payoffs, the first player chooses R. Hence the outcome of the game can be predicted as [5,4,1].

# Example (cont.)

Strategies identified by the sequential rationality principle (and *backward induction*) are:

$s_1$ = R,

$s_2$ = A if '1 plays R',

$s_3$ = D if '1 plays L' or if '1 plays R and 2 plays A',

The Nash equilibrium is (R,A,D)

# (T.19) Zermelo Theorem

Every finite game with perfect information $\Gamma_E$ has a Nash equilibrium in pure strategies which can be found using backward induction. Moreover if none of the players has identical payoffs in two different terminal nodes, this is the only Nash equilibrium that can be found in this way.

(easy to prove)

# Questions:

Q-8 A game in an extensive form (D.26)
[a] <u>captures not only outcomes but also ways the players followed in order to achieve them</u>
[b] requires that in a successor node the player moves who did not move in the predecessor node
[c] does not contain information sets consisting of an odd number of nodes
[d] may have nodes that can be reached by more than one action
[e] none of the above

# Exercises:

E-8 Discuss the Zermelo theorem without the assumption of the lack of identical payoffs. What happens if in two terminal nodes $x_1$ and $x_2$ there are identical payoffs for the player who does not move in the node $p(x_1)=p(x_2)$?

# **Players do not move simultaneously, and they remember all their previous moves** (cont.)

Subgame perfection

(D.30)

Subgame of an extensive form game $\Gamma_E$ – a subset satisfying the following two conditions:

1. It starts with an information set consisting of a single decision node x, contains all subsequent nodes, s(x), s(s(x)) and so on, and does not contain other nodes;
2. If the node x is in the subgame then every $x' \in H(x)$ is in it as well (i.e. a subgame does not contain 'incomplete' information sets).

<u>(T.20)</u>

The entire game $\Gamma_E$ is also a subgame

(easy to prove)

<u>(T.21)</u>

Every decision node in a finite game with perfect information can 'start' a subgame

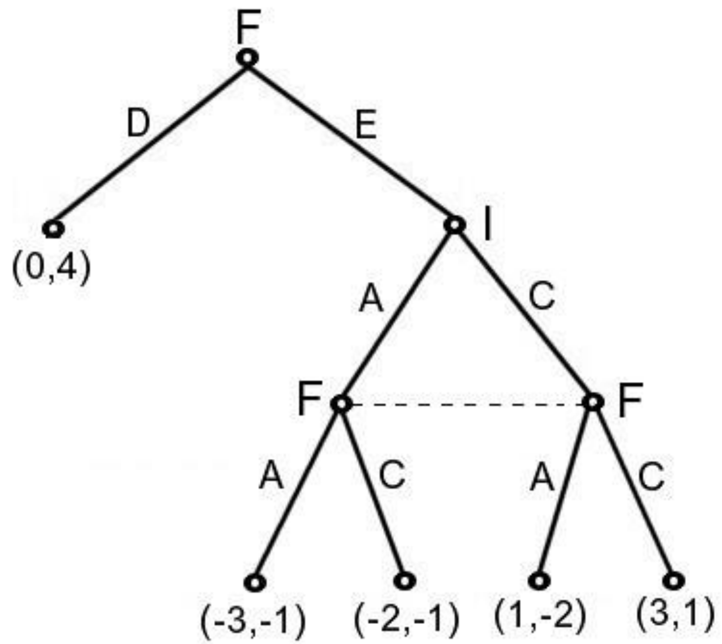
(easy to prove)

# Example I

A market is served by one firm called Incumbent (I). Another firm (F) contemplates entering the market. It has two strategies: Enter (E) and Do Not Enter (D). If it chooses D, the payoffs are [0,4] (its payoff is 0, and the payoff of the Incumbent is – as before – 4). If it chooses E, the two firms will play a duopolistic game which boils down either to attacking (A) e.g. by a price war or to cooperating (C), e.g. à la Cournot or Stackelberg (or by sharing the market). If both attack then the payoffs are [-3,-1], if they cooperate then they are [3,1].

# Example I (cont.)

If I attacks and F cooperates it is [-2,-1], and if I cooperates and F attacks it is [1,-2]. The story can be reflected by a game $\Gamma_E$ defined by the tree:



# Example I (cont.)

If F chooses E, then I has two strategies: A and C. Choosing its own strategy, F does not know what strategy has been chosen by I. This is reflected by the two nodes where F moves belonging to the same information set (the two nodes are linked by a dotted line). The game $\Gamma_E$ has two subgames: the entire game and, say, $\Gamma_O$ the oligopolistic rivalry part following the decision E taken by F.

## Example I (cont.)

By T.21, the number of subgames is the difference between the number of non-terminal nodes (here 4) and the number of nodes 'hidden' in non-single node information sets (here 2). Hence the number of subgames is 2. Since by T.20 $\Gamma_E$ is also its subgame, then $\Gamma_O$ is the only other subgame.

# Example I (cont.)

The game $\Gamma_E$ has the following strategic (normal) form $\Gamma_N$:

| | | Incumbent (I) | |
|---|---|---|---|
| | | C if F enters | A if F enters |
| Entering Firm (F) | D&C | 0,4 | 0,4 |
| | D&A | 0,4 | 0,4 |
| | E&C | 3,1 | -2,-1 |
| | E&A | 1,-2 | -3,-1 |

# Example I (cont.)

This strategic-form game has three Nash equilibria (D&C,A), (D&A,A), and (E&C,C) with payoffs (0,4), (0,4), and (3,1), respectively. The first two do not satisfy the sequential rationality principle (I's threat that it will attack if F enters is not credible).

# Example I (cont.)

The subgame $\Gamma_O$ has the following strategic form (payoff matrix):

<table>
<tr><td colspan="2" rowspan="2"></td><td colspan="2">Incumbent (I)</td></tr>
<tr><td>C</td><td>A</td></tr>
<tr><td rowspan="2">Entering Firm (F)</td><td>C</td><td>3,1</td><td>-2,-1</td></tr>
<tr><td>A</td><td>1,-2</td><td>-3,-1</td></tr>
</table>

The only Nash equilibrium in this subgame is (C,C).

<u>(D.31)</u>
A strategy profile $s=(s_1,\ldots,s_n)$ in an n-person game $\Gamma_E$ induces a Nash equilibrium in a subgame of this game, if actions defined by s for information sets of this subgame (understood as a separate game) make a Nash equilibrium in it

<u>(D.32)</u> A strategy profile $s=(s_1,\ldots,s_n)$ in an n-person game $\Gamma_E$ is called <u>Subgame Perfect Nash Equilibrium</u>, SPNE, if it induces a Nash equilibrium in its every subgame

(T.22)

Every finite game with full information $\Gamma_E$ has an SPNE in pure strategies. Moreover, if none of the players has identical payoffs in two different terminal nodes, this is the only SPNE

(easy to prove)

Corollary (from D.32, not T.22):

In the example I above (E&C,C) is an SPNE.

Warning:

Chess satisfies the assumptions of T.22

Note:

Neither of the Nash equilibria that were not SPNEs in the example I above seemed realistic, since the threat of choosing A by I (if F enters) was not credible. The potential entrant may suspect that when confronted with its presence in the market – despite threats – the incumbent will cooperate. Thus such an entry deterrence strategy of the incumbent is not effective.

Note:

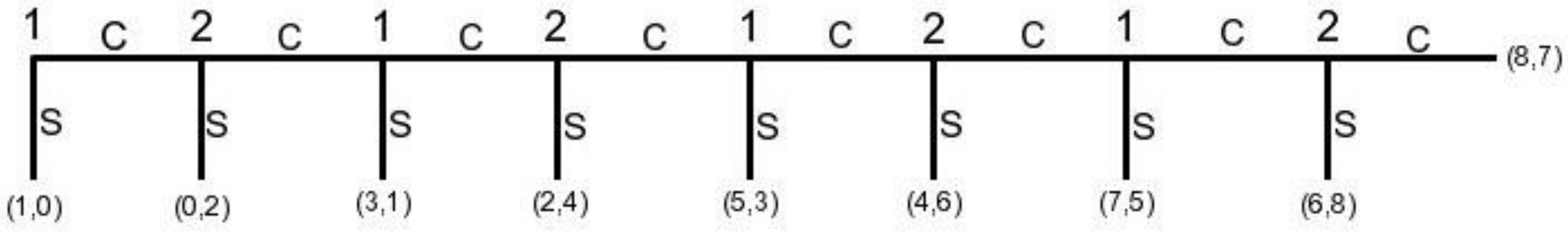
To be credible, a threat must include a decision that will be taken irrespectively of the 'sequential rationality principle'. This is called a pre-commitment. Examples of pre-commitment include: (in military planning) automatic destruction of one's own infrastructure in the case of attack, and (in economics) lack of possibility of changing a business decision before a certain date.

# Example II (Centipede game)

This 2-person game lasts 2k periods (where k=1,2,3,...), it has 2k decision nodes, and 2k+1 terminal nodes. The players move in turns, starting with the player number 1. In every node either of the two decisions can be taken: Stop (S) or Continue (C). If the game is stopped at node $i=1,2,3,...,2k$ then the payoffs are: $P(i)=i-(1+(-1)^i)$, $D(1)=0$, and for $i>1$ $D(i)=i-(1-(-1)^i)$. If the game is not stopped by any player at any node, it terminates with the payoffs $P(2k)=2k$, and $D(2k)=2k-1$.

# Example II (cont.)

For k=4 the game is reflected by the following tree (the name refers to the shape of the tree):



# Example II (cont.)

At any stage (node) each player receives a higher payoff if he/she stops the game than if the game is stopped by the other player in the next node (stage). By the sequential rationality principle, player number 2 is better off if he/she stops the game in its ultimate (*2k*th) decision node. Knowing this, player number 1 has an incentive to stop the game in its penultimate (*(2k-1)*th) decision node. Knowing this, player number 2 has an incentive to stop the game earlier, i.e. in the *(2k-2)*th node, and so on. Therefore by the sequential rationality principle, the game should be stopped in its first node.

# Example II (cont.)

Experiments do not confirm the outcome predicted by the sequential rationality principle in this case. Hence in real life people's behaviour is more complicated than some game theoretic models assume.

# Questions:

Q-9 The concept of a Subgame Perfect Nash Equilibrium in finite 2-person games
[a] requires all subgames to contain only one-node information sets
[b] eliminates Nash equilibria which violate the sequential rationality principle
[c] cannot be extended to n-person games (where n>2)
[d] in the market entry-deterrence game allows for strategies involving non-credible threats
[e] none of the above

# Exercises:

E-9 In the entry-deterrence game described in $\Gamma_N$ above (Example I), please modify the payoffs understood as profits so as to account for the following market conditions. The market price is p=a-by, where the supply y comes from the incumbent (I) or the incumbent and the entrant (F). Both firms have identical cost functions and constant returns to scale, i.e. $MC_I=MC_F=AC_I=AC_F=c=const$. Cooperation means setting the supply level as in the Cournot model. Attack means setting individual supplies at the level which maximizes profit for a single firm (as if it were a monopolist).

# Repeated games I

Example I (Sequential Bertrand model)

A duopoly plays a Bertrand game sequentially. After each period the rivals j=1,2 can (simultaneously) announce new prices. It is assumed that they maximize the Present Value (with the discount rate $\rho>0$) of their profits:

$$\pi_{j0}+\pi_{j1}/(1+\rho)+\pi_{j2}/(1+\rho)^2+\pi_{j3}/(1+\rho)^3+\ldots$$

The game can be finite (if firms play it T times) or infinite.

Discounting (recollection)

Discount rate ($\rho$) lets compare money amounts that belong to different time periods

$$X_t = X_0(1+r)^\rho, \text{ or } X_0 = X_t/(1+r)^\rho, \text{ where}$$

$X_t$ is the present value (in year t) of the value $X_0$ observed in year 0; or $X_0$ is the present value (in year 0) of the value $X_t$ observed in year t.

Present value of quantities $X_0$, $X_1$, $X_2$, …, $X_T$:

$$PV = X_0/(1+\rho)^0+X_1/(1+\rho)^1+X_2/(1+\rho)^2+...+X_T/(1+\rho)^T.$$

(T.23)

In the sequential Bertrand model, if the game is finite, then there is only one SPNE; it consists of Nash strategies from a single Bertrand model, i.e. (c,c), where $c=AC_1=AC_2=MC_1=MC_2=const$ (as in the standard Bertrand duopoly model)

Proof:

No price $p<c$ can be sustained since it implies losses. Yet no price $p>c$ can be sustained since at least one firm has a motivation to offer a price $\in(c,p)$ in order to undercut the rival's price, get all the clients and increase the profit.

## (D.33) Nash-Bertrand reversion strategy

In the sequential Bertrand model we define for t>1

$$H_{t-1}=\{(p_{11},p_{21}),(p_{12},p_{22}),(p_{13},p_{23}),\ldots,(p_{1t-1},p_{2t-1})\}$$

i.e. a history of strategies applied. In the *t*th stage the rivals choose $p_{jt}(H_{t-1})=p^M$ if t=1 or $H_{t-1}$ consists of $(p^M,p^M)$ only. Otherwise $p_{jt}(H_{t-1})=c$. $p^M$ is the monopolistic price, i.e. the price giving the two firms the maximum profit available jointly for the suppliers. In the standard model (two identical firms as in T.23, linear demand curve p=a-by) the monopolistic price $p^M=(a+c)/2$. The joint supply is $y^M=(a-c)/(2b)$, and the joint profit is $\pi^M=(a-c)^2/(4b)$.

<u>(T.24)</u>

In the sequential Bertrand model, if the game is infinite, then Nash-Bertrand reversion strategy is SPNE if and only if $\rho \leq 1$

(easy to prove)

<u>(T.25)</u>

In the sequential Bertrand model, if the game is infinite and $\rho \leq 1$, then each fixed and common (for both players) selection of the price $p \in [c, p^M]$ backed by the Nash-Bertrand reversion strategy is an SPNE.

(easy to prove)

<u>(T.26)</u>

In the sequential Bertrand model, if the game is infinite and $\rho>1$, then the only SPNE consists of $p=c$ to be selected by both players in each stage.

(easy to prove)

Example II (Sequential Cournot model)

A duopoly plays a Cournot game sequentially. After each period the rivals j=1,2 can (simultaneously) announce new supplies. It is assumed that they maximize the Present Value (with the discount rate $\rho>0$) of their profits:

$$\pi_{j0}+\pi_{j1}/(1+\rho)+\pi_{j2}/(1+\rho)^2+\pi_{j3}/(1+\rho)^3+\ldots$$

The game can be finite (if firms play it T times) or infinite.

(T.27)

In the sequential Cournot model, if the game is finite, then there is only one SPNE; it consists of Nash strategies from a single Cournot model, i.e. $(y^C, y^C)$, where in the standard Cournot duopoly ($c=AC_1=AC_2=MC_1=MC_2=const$) with a linear demand curve $p=a-by$, $y^C=y_1=y_2=(a-c)/3b$. The total supply is thus $y=2(a-c)/(3b)$, the price $p^C=(a-2c)/3$, and the joint profit $\pi^C=(a-c)^2/(9b)$.

(easy to prove)

## (D.34) Nash-Cournot reversion strategy

In the sequential Cournot model we define for t>1

$$H_{t-1}=\{(y_{11},y_{21}),(y_{12},y_{22}),(y_{13},y_{23}),\ldots,(y_{1t-1},y_{2t-1})\}$$

i.e. a history of strategies applied. In the *t*th stage the rivals choose $y_{jt}(H_{t-1})=y^M/2$if t=1 or $H_{t-1}$ consists of $(y^M/2,y^M/2)$ only. Otherwise $p_{jt}(H_{t-1})=y^C$. $y^M/2$ is the half of the monopolistic supply, i.e. the (joint) supply giving the two firms the maximum profit available jointly for the suppliers. In the standard model (two identical firms as in T.27, linear demand curve p=a-by) the monopolistic joint supply is $y^M=(a-c)/(2b)$, and the resulting price $p^M=(a+c)/2$. The joint profit is then $\pi^M=(a-c)^2/(4b)$.

<u>(T.28)</u>

In the sequential Cournot model, if the game is infinite, then Nash-Cournot reversion strategy is SPNE if $\rho \leq 8/9$

(difficult to prove)

<u>(T.29)</u>

In the sequential Cournot model, if the game is infinite and $\rho > 8/9$, then the only SPNE consists of $y = y^C$ to be selected by both players in each stage.

(difficult to prove)

'Folk theorems' – useful theorems that everybody relies on, but so trivial that nobody bothers to prove and/or authorize with one's name. Several game theory results are nick-named 'folk theorems'.

Folk theorem (example)

In a finite game, by backward induction one can prove that the only SPNE consists of non-cooperative decisions

## Note

As the difference between an infinite game and a finite one with a large number of stages is difficult to appreciate in an experimental setting, while the consequences are drastically different (as evident by comparing the Folk Theorem above with previous theorems), economists introduce the concept of finite games of unknown duration.

# Note

In examples I and II, 'Nash reversion strategies' involved 'punishing' the non-cooperating rival by switching to a Nash (non-cooperative) solution for ever. An alternative concept is to 'punish' the non-cooperating rival only once by choosing a Nash solution and then moving back to a cooperative one from the next stage on. There are empirical findings which indicate that this other 'Nash reversion strategy' provides higher payoffs for the players.

# Tit-For-Tat

- Robert Axelrod tournament

- Anatol Rapoport:
  'Tit for tat' = a punishment for the non-cooperative behaviour, but not for ever – only once.

## Note

In both examples, there is no collusion between the rivals. Nevertheless their choices are similar and may create an impression that they colluded. That is why this type of behaviour is sometimes referred to as 'tacit collusion'.

# Questions:

Q-10 The 'Nash reversion strategy' (as defined in D.33 or D34)
[a] means that rivals agree to play their respective Nash strategies
[b] <u>means that rivals are forced to cooperate by a threat that non-cooperation would be punished by slipping into a Nash equilibrium</u>
[c] implies that rivals 'revert' strategies by playing each other roles
[d] applies to finite games only
[e] none of the above

# Exercises:

E-10 Please design an experiment to let agents avoid the non-cooperative behaviour implicit in the Folk Theorem for any finite game. Provide a detailed explanation of motivations the agents may or may not have to cooperate during the experiment.

# Repeated games II

Refining the concept of Nash equilibrium in a dynamic setting; 'bygone is bygone'

## (D.35) Pareto domination

A pair of strategies $(i_1,j_1)$ is Pareto-dominated by $(i_2,j_2)$, if $P_{i2j2} \geq P_{i1j1}$ and $D_{i2j2} \geq D_{i1j1}$. In other words, if players are free to choose and rational they should avoid playing Pareto-dominated strategies.

## (D.36) Pareto perfection (informal)

'Players are not likely to choose strategies that are Pareto-dominated, irrespective of previous choices'

## Note

Pareto perfection is also called 'renegotiation proofness', as – irrespective of previous (non-binding) agreements – players are unlikely to stick to their threats (such as "Nash reversion strategy") if a Pareto superior option is available at a given stage

# <u>(D.37) Markov Perfection</u> (informal)

In a repeated game, payoffs depend only on a state variable (not on the history of specific strategies)

<u>Example I</u> (Extraction of a common resource)

Two players, 1 and 2, exploit the stock of a common resource $K^t \geq 0$. In each period t=1,2,3,... they plan how much of the stock to extract $a^t_1$, $a^t_2$. If $a^t_1+a^t_2<K^t$, then their actual extraction quantities are $s^t_i=a^t_i$ for i=1,2. If $a^t_1+a^t_2 \geq K^t$ then $s^t_i=K^t/2$ for i=1,2. Their payoffs are $\pi_i(s^t_i)$ for i=1,2. The stock of the resource is $K^{t+1}=f(K^t-s^t_1-s^t_2)$. If the stock is depleted completely then it will never be created again: $f(0)=0$.

## Example I (continued)

Markov-perfect strategies $(s^1_1,s^1_2)$, $(s^2_1,s^2_2)$, $(s^3_1,s^3_2)$, ... depend only on $K^1$, $K^2$, $K^3$, ... and not on the history of earlier decisions (of either player). Critics say that it would be difficult to find systems which comply with this assumption. Perhaps it is more realistic to assume that players retain some memory of how they behaved in the past.

# D.38 Trembling-hand-perfect Nash equilibrium (THPNE) (almost formal definition)

THPNE is a Nash equilibrium that is robust to small perturbations. A strict mathematical definition is much more complex and it will not be quoted here. Instead an example will be analyzed to explain which Nash equilibria are robust enough (are THPNE), and which are not.

## Example II

In the following game there are two Nash equilibria: (U,L) and (D,R).

| | | Second | |
|---|---|---|---|
| | | L | R |
| First | U | (1,1) | (2,0) |
| | D | (0,2) | (2,2) |

However, only (U,L) is THPNE, as the following argument explains.

Example II (continued)

As (U,L) is a Nash equilibrium, the players are likely to choose pure rather than mixed strategies. In other words, the First is likely to play U and D with probabilities 1 and 0, respectively. Likewise the Second is likely to play L and R with probabilities 1 and 0, respectively. Neither has an incentive to unilaterally change the decision. Now let us assume that they make mistakes occasionally (their hands 'tremble'). I.e. let us assume that the First chooses U and D with probabilities 1-$\varepsilon$ and $\varepsilon$, respectively (if the hand does not 'tremble', $\varepsilon$=0). What is the optimum mixed strategy of the Second player?

Example II (continued)

If the Second plays L then the expected payoff will be $1(1-\varepsilon)+2\varepsilon=1+\varepsilon$. If the Second plays R then the expected payoff will be $0(1-\varepsilon)+2\varepsilon=2\varepsilon$. Thus, for small $\varepsilon$, the Second will maximize the expected payoff by choosing L with high probability and R with low probability. Now let us assume that the hands of the Second player tremble, but the First one never fails to choose an appropriate strategy.

## Example II (continued)

Thus, let us assume that the Second player chooses L and R with probabilities 1-$\varepsilon$ and $\varepsilon$, respectively, then the First will maximize payoff by preferring U rather than D. Hence (U,L) turns out to be a Nash equilibrium robust with respect to such perturbations, i.e. a THPNE.

<u>Example II</u> (continued)

Now let us analyze (D,R). Assuming that the First player chooses U and D with probabilities ε and 1-ε, respectively, the Second will get the expected payoff of $1\varepsilon+2(1-\varepsilon)=2-\varepsilon$ (if playing L) or $0\varepsilon+2(1-\varepsilon)=2-2\varepsilon$ (if playing R). Therefore the Second will choose L more frequently. By symmetry of payoffs, the First will choose more frequently U when the Second plays L and R with probabilities ε and 1-ε, respectively (if the hands of the Second player tremble, but the First one never fails). Either calculation explains that (D,R) is not THPNE. All arguments can be made mathematically more precise, but this is not required if the concept of THPNE is only intuitively defined (as it is in this course).

# Questions:

Q-11 'Trembling-hand' perfection
[a] is a method to identify strategies that do not comply with the definition of Nash equilibrium
[b] applies only to games that do not have Nash equilibria in pure strategies
[c] assumes that players never fail to choose optimum strategies
[d] assumes that players care not only for their own outcomes but also for the total payoff
<u>[e] none of the above</u>

# Exercises:

E-11 Consider a common resource game such as in the Example I above with the following specifications. Two users exploit a renewable resource whose regeneration rate is given by the so-called logistical equation: if $s^t_1+s^t_2<K^t$ then $K^{t+1}=K^t+gK^t(K-K^t)-s^t_1-s^t_2$, where g,K>0; please note that if t=1,2,3,... number consecutive years then: (1) $K^{t+1}=K^t$ when $gK^t(K-K^t)=s^t_1+s^t_2$ (i.e. when total extraction is equal to the annual increment $\Delta K^t=gK^t(K-K^t)$); (2) the annual increment $\Delta K^t=0$ in two cases: for $K^t=0$ and for $K^t=K$; the former case simply states that if there is no stock, it will not be replenished; the latter lets interpret K as the 'carrying (maximum) capacity'); (3) $\Delta K^t$ is largest if $K^t=K/2$. Let us assume further that $\pi_i(x)=x$ for i=1,2 which means that the stock can be sold at a unit price and the marginal cost of extraction is zero. Let $K^0=K/2$. If the players choose $a^t_1=a^t_2=\Delta K^t/2$ then they will continue the extraction for ever and every year they will jointly enjoy the largest payoffs possible. What are game theoretic predictions for the game?

# Coalitions and general equilibrium I

- Trivial coalitions in the case of two-person games
- Edgeworth box used to illustrate pure exchange games

<u>(D.39)</u>

Let there be I players {1,...,I} as in D.20. A coalition $K \subset \{1,\ldots,I\}$ <u>improves upon</u>, or <u>blocks</u>, the outcome $(s^*_1,\ldots,s^*_I)$ if there is a set of strategies $(s_1,\ldots,s_I)$ such that for every $i \in K$ $u_i(s_1,\ldots,s_I) > u_i(s^*_1,\ldots,s^*_I)$.

<u>(D.40)</u>

The outcome $(s^*_1,\ldots,s^*_I)$ has the <u>core property</u>, if there is no coalition that can improve upon it. The set of outcomes that have the core property is called the <u>core</u>.

## Example I

The 'Prisoner's Dilemma' game given by the following payoff matrix

| | | Second | |
|---|---|---|---|
| | | C | D |
| First | C | (-12,-12) | (0,-18) |
| | D | (-18,0) | (-1,-1) |

has an empty core. It can be seen that for every outcome (cell in the matrix) at least one player can make a coalition (with himself or herself) to improve his/her payoff

<u>Note</u>

Definitions D.39 and D.40 can be extended to games with infinite number of strategies.

<u>(D.41)</u>

A <u>coalitional game with transferable payoffs</u> is defined as <N,v>, where N is the finite set of players, and v is a real function defined over the set of all possible coalitions $K \subset N$. v(K) is called the <u>worth of the coalition</u>. v(N) is the <u>worth of the game</u>. If $x_1,\ldots,x_n$ are payoffs then $x(K)=\sum_{i \in K} x_i$ is called a <u>payoff profile</u>. A payoff profile is <u>feasible</u> if $x(K)=v(K)$. By convention, $v(\emptyset)=0$.

(D.42)

Shapley value of a coalitional game with transferable payoffs <N,v> for the player i is:

$$Sh_i(N,v) = (1/(\mathrm{card}N)!)\sum m(K(\pi,i),i)$$

where the summation extends over all permutations of the set N. The number $m(K(\pi,i),i)$ is the increase of the sum of payoffs the player i 'brings' to a coalition $K(\pi,i)$ consisting of those members of the set N who in the permutation $\pi$ precede i.

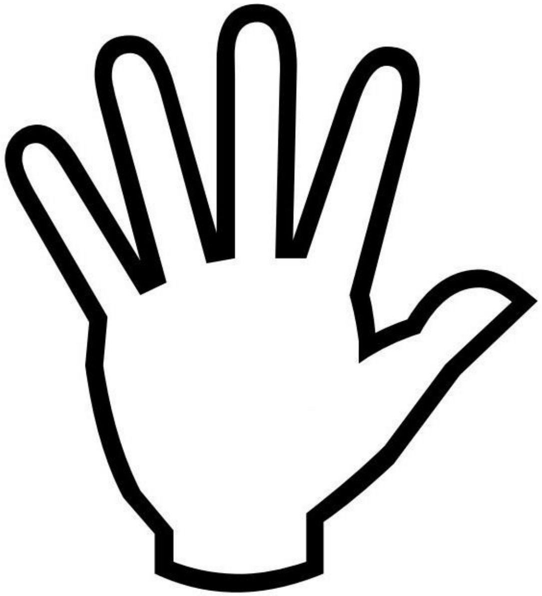
## Note

Elements of Shapley value can be interpreted as the threat of leaving made by the player i to a coalition he/she is a member of (K) consisting of players whose place precedes i:

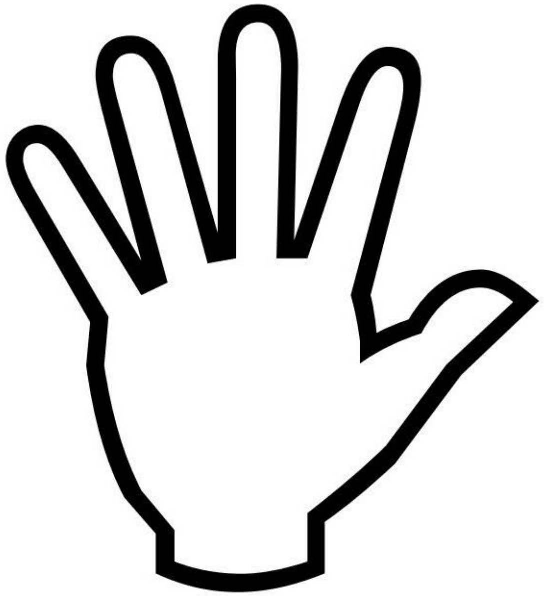
$$(v(K)-v(K\setminus\{i\})$$

The sum of these numbers extends over all possible orderings (permutations) of the members of N. Thus $Sh_i(N,v)$ can be also interpreted as an expected value the player i is 'entitled to' if other players behave rationally.

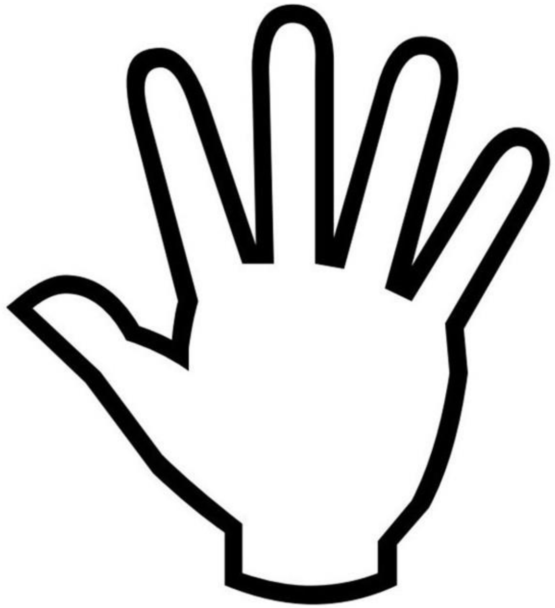
<u>Example II</u> (left and right hand gloves):



1



2



3

# Example II (cont.)

| Ordering | 1 | 2 | 3 |
|---|---|---|---|
| {1,2,3} | 0 | 0 | 12 |
| {1,3,2} | 0 | 0 | 12 |
| {2,1,3} | 0 | 0 | 12 |
| {2,3,1} | 0 | 0 | 12 |
| {3,1,2} | 12 | 0 | 0 |
| {3,2,1} | 0 | 12 | 0 |
| $\Sigma$ | 12 | 12 | 48 |

# Example II (cont.)

- $Sh_1=12/6=2$,
- $Sh_2=12/6=2$, and
- $Sh_3=48/6=8$.

$$Sh_1 + Sh_2 + Sh_3 = 2+2+8 = 12$$

# Example III (capitalist production)

A capitalist owns a factory and each of w workers owns his/her labour only. Without the capitalists workers cannot produce anything. Any group of m workers (m>0) produce f(m), where f: $\Re \rightarrow \Re$ is a linear function f(m)=mp (note that f(0)=0) with p>0 interpreted as the marginal and average productivity of a worker (the same for any worker).

# Example III (continued)

This can be modelled as the following coalitional game with transferable payoffs <N,v>. $N=\{c\}\cup W$, where c denotes the capitalist and W – the set of workers. The game is worth

- $v(K)=0$ if $c\notin K$, and
- $v(K)=f(\mathrm{card}(K\cap W))$ if $c\in K$

# Example III (continued)

The core of this game is

$$\{x \in \Re^{\mathrm{card}N} : 0 \leq x_i \leq p \text{ for } i \in W \text{ and } \textstyle\sum_{i \in N} x_i = wp\},$$

where w is the number of workers participating.

Its Shapley value is

- $Sh_c(N,v)=wp/2$, and
- $Sh_i(N,v)=p/2$

Note that the core includes [wp/2,p/2,...,p/2], but it also includes the 'competitive' outcome [0,p,...,p].

## Example IV (A three-player majority game)

Three players can obtain a payoff of 1; any two of them can obtain jointly $\alpha \in [0,1]$ irrespective of the actions of the third; each player alone can obtain nothing irrespective of the actions of the remaining two.

# Example IV (continued)

This is a coalitional game with transferable payoffs <N,v> where $N=\{1,2,3\}$, $v(N)=1$, $v(K)=\alpha$ whenever card K =2, and $v(\{i\})=0$ whenever $i=1,2,3$. The core of this game is the set of all nonnegative payoff vectors $[x_1,x_2,x_3]$ such that $x(N)=1$ and $x(K)\geq\alpha$ for every two-player coalition. Therefore if $\alpha>2/3$ the core is empty. (Assuming that all three players are in coalition (K=N), if one of them gets $x_i\geq 1/3$ then the other two can make a coalition to get jointly at least 2/3, i.e. more than before.)

# Example V (Miners game)

I>2 miners discovered heavy identical pieces of a valuable mineral. Each piece can be carried out (and privately sold for the price of 1, e.g. ½ for each of the carrying miner) by two miners. Will they agree on how to handle this discovery?

# Example V (continued)

The problem can be modelled as a coalitional game with transferable payoffs <N,v>, where

- v(K)=(card K)/2 if card K is even; and
- v(K)=((card K)-1) if card K is odd.

If I≥4 and even then the core consists of a single payoff profile [1/2,...,1/2].

If I≥3 and odd then the core is empty.

# Questions:

Q-12 The 'Prisoner's Dilemma' game (Example I) has an empty core
[a] because the players do not cooperate with each other
<u>[b] because at least one player can always improve his/her payoff</u>
[c] because both players understand that the game will not be repeated
[d] because the players will lose even if they cooperate
[e] none of the above

# Exercises:

E-12 Let us assume that three-players majority game is played with $\alpha$=4/5 (example IV above). The payoff profile is not symmetric: [1/4,1/4,1/2]. One of the players who gets 1/4 approaches the other one who gets 1/4 suggesting that once they get rid of the third player (who gets 1/2) they will get $\alpha$=4/5 to be split into halves (2/5 each), so that both of them can be better off. Does this offer make sense?

# Coalitions and general equilibrium II

## Note: Pure exchange economy

A so-called pure exchange economy can be considered a game. In what follows we will confine to the case where two consumers hold two goods and contemplate whether to exchange some units of one good for some units of the other one. The players consider only allocations which do not make them worse off.

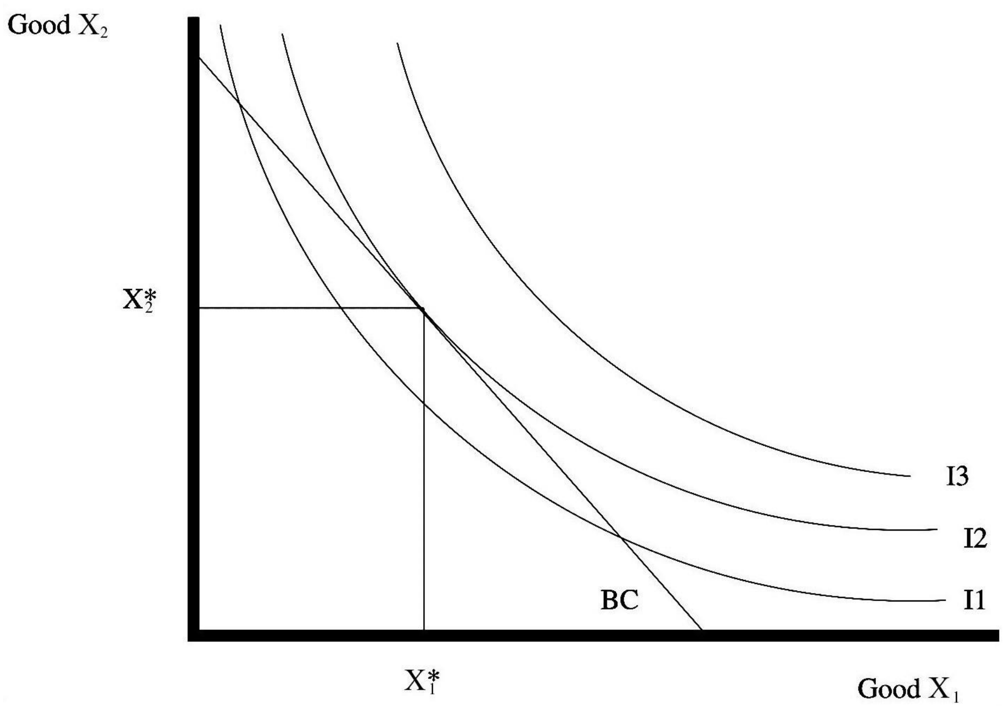
# Note (continued)

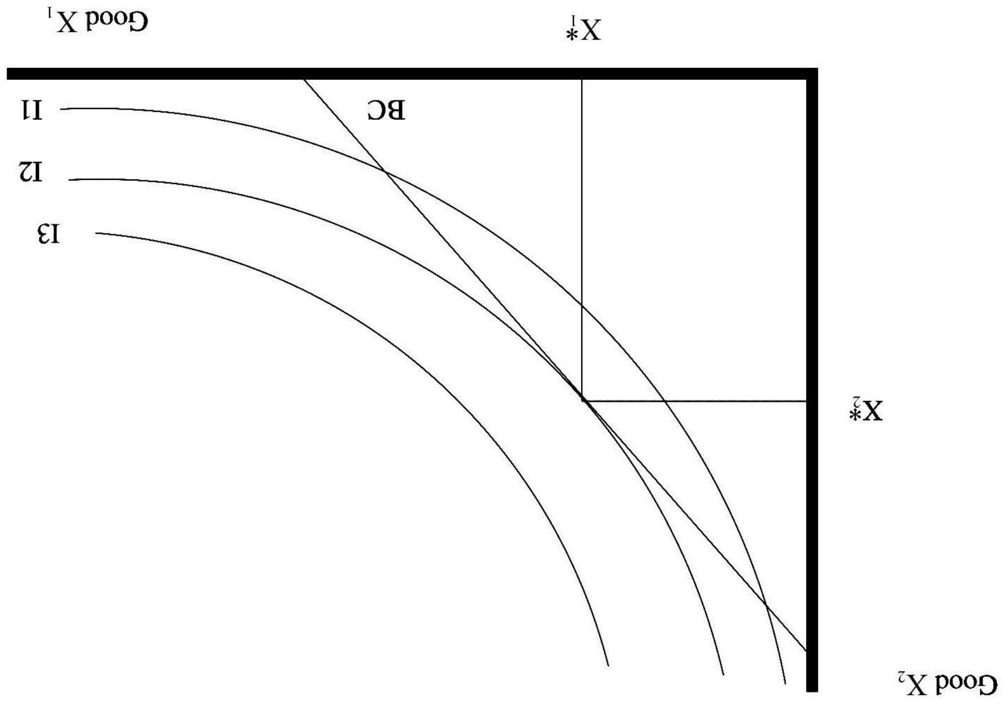
- Endowment (initial allocation) of the *i*th consumer: $\omega_{i1}$, $\omega_{i2}$
- Total endowment of the *j*th commodity: $\omega_j = \omega_{1j}+\omega_{2j}$
- Gross demand (final allocation) of the *i*th consumer: $x_{i1}, x_{i2}$
- Total demand for the *j*th commodity: $x_j = x_{1j}+x_{2j}$
- Excess demand of the *i*th consumer: $x_{i1}-\omega_{i1}, x_{i2}-\omega_{i2}$

# Note: Edgeworth box

A graphical analysis of feasible allocations in a pure exchange economy (superposition of two coordinate systems for the analysis of a consumer's choice: the width of the rectangle = $\omega_{11}+\omega_{21}$, the height of the rectangle = $\omega_{12}+\omega_{22}$; the second system is rotated by 180º)

# Edgeworth box idea

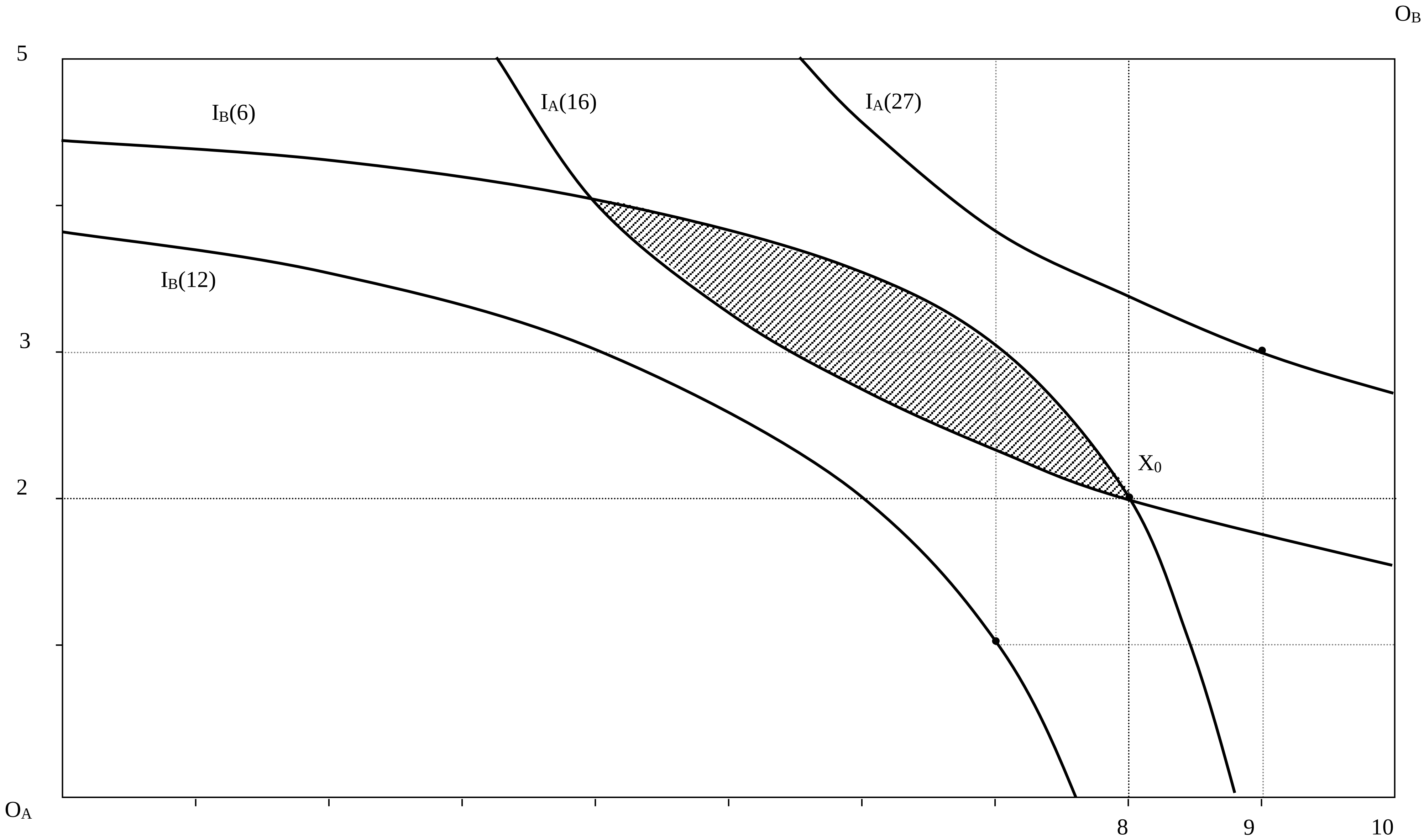


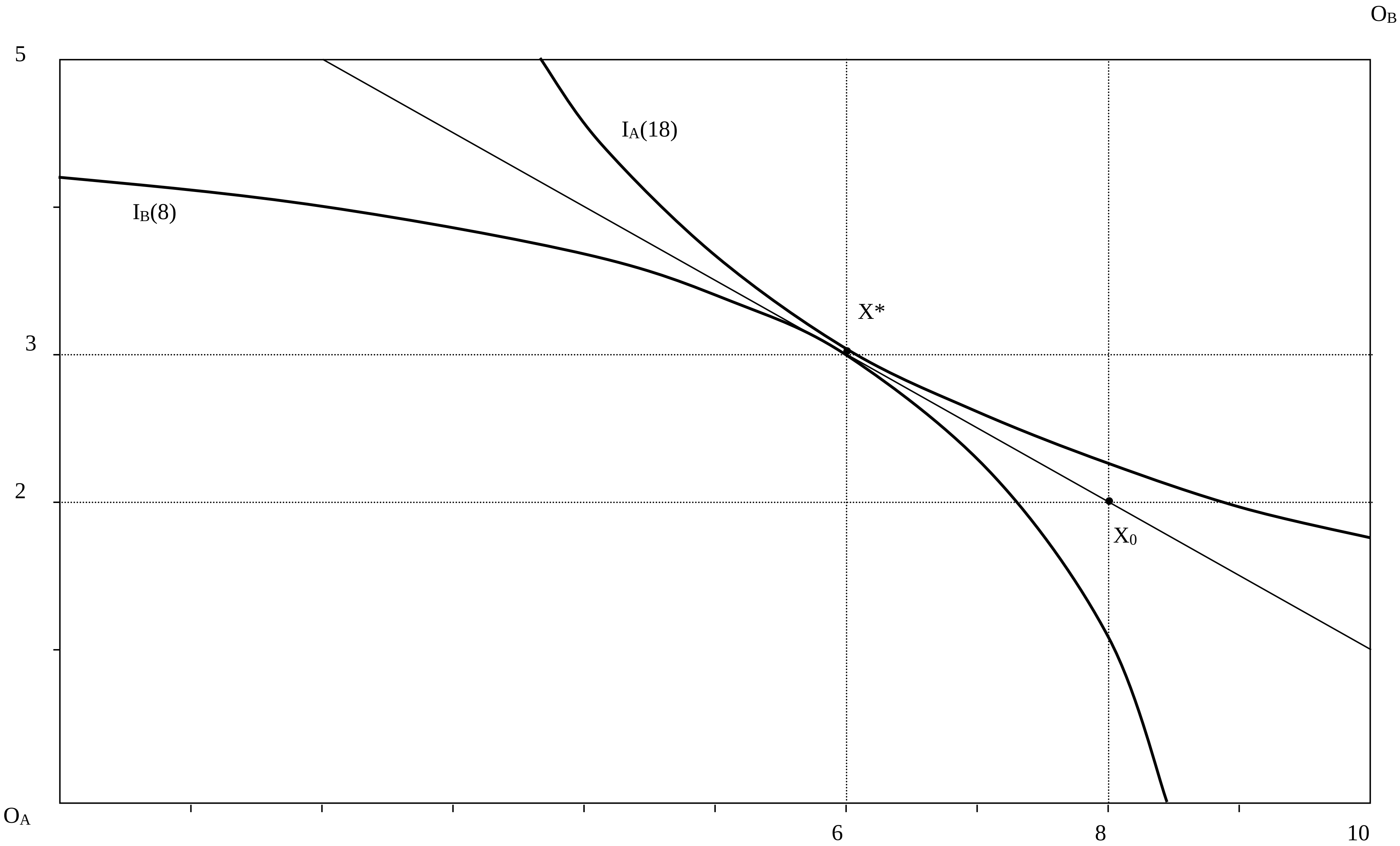
Interpretation of the box.

- Good number 1 = apples,
- Good number 2 = oranges.
- The first person (A) brought 8 apples and 2 oranges,
- The second person (B, whose axes were rotated by 180$^{o}$) brought 2 apples and 3 oranges.
- They have 10 apples and 5 oranges jointly.

GT-13-6



GT-13-7



# Note: Edgeworth box (continued)

Pictures illustrate the following situation. Indifference curves (i.e. the sets of points yielding the same utility) of A, $I_A(\alpha)$ are given by the formula $x_{2A}=\alpha/x_{1A}$, while indifference curves of B, $I_B(\beta)$ are given by $x_{2B}=\beta/x_{1B}$ ($\alpha,\beta>0$ – parameters); additionally, we assume that the total quantity of the first good is 10, while that of the second – 5. Moreover, the diagram corresponds to the initial allocation of the first good 8:2, while of the second one – 2:3 (point $X_0$). There are two indifference curves containing this point:

# Note: Edgeworth box (continued)

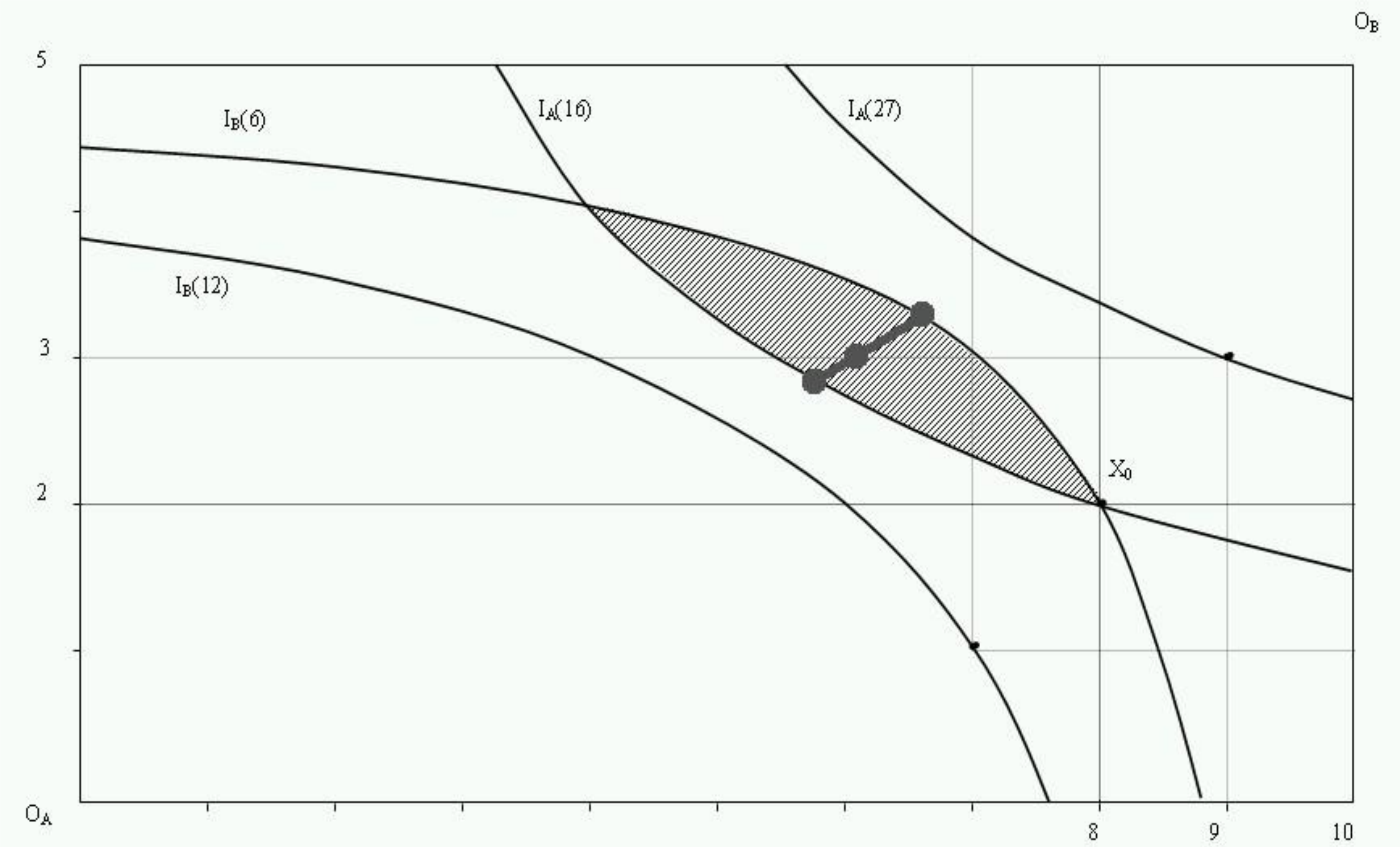
$x_{2A}=16/x_{1A}$ ($\alpha=16$) and $x_{2B}=6/x_{1B}$ ($\beta=6$). A would prefer to be on a higher indifference curve, say, in (9,3), i.e. on the curve $x_{2A}=27/x_{1A}$ ($\alpha=27$). At the same time, B would like to have more of everything, i.e. to be in, say, (3,4), i.e. on the curve $x_{2B}=12/x_{1B}$ ($\beta=12$). It is impossible to satisfy these expectations at the same time. One solution which can place both agents in a jointly preferred point (one should solve a system of simultaneous equations) is: $x_{1A}*=6$, $x_{2A}*=3$, $x_{1B}*=4$, $x_{2B}*=2$, $\alpha=18$, $\beta=8$, $p= p_1/p_2 =0,5$

# Note: Edgeworth box (continued)

(see the second picture). Agents A and B are on $I_A(18)$ and $I_B(8)$, respectively, and they are better of than in $X_0$. One can see from the figure that they cannot improve their situations further simultaneously. In other words, (6,3) is a Pareto optimum. Equilibrium prices which satisfy this solution are multiple, e.g. $p_1=1$, $p_2=2$, or $p_1=7$, $p_2=14$, or $p_1=0{,}5$, $p_2=1$ etc, as long as $p_1/p_2=p=0{,}5$.

# Note: Edgeworth box (continued)

A contract curve is the set of all allocations which are Pareto optima and which are preferred by both players over their initial allocation. It is depicted as a thick line in the following picture. A contract curve in an Edgeworth box is a core in a pure exchange economy game.


O$_B$
5
I$_B$(6)
I$_A$(16)
I$_A$(27)
I$_B$(12)
3
X$_0$
2
O$_A$
8
9
10

(D.43)

$X^*$ is a Walrasian (competitive) equilibrium in an Edgeworth box pure exchange economy if:

- $u_A(x^*_{1A},x^*_{2A}) \geq u_A(x_{1A},x_{2A})$ for all $(x_{1A},x_{2A}) \in B_{\mathbf{p}}(X_0)$,
- $u_B(x^*_{1B},x^*_{2B}) \geq u_B(x_{1B},x_{2B})$ for all $(x_{1B},x_{2B}) \in B_{\mathbf{p}}(X_0)$,
- $B_{\mathbf{p}}(X_0)=\{(x_{1A},x_{2A},x_{1B},x_{2B}) \in \Re^4$:

$$p_1x_{1A}+p_2x_{2A} \leq p_1\omega_{1A}+p_2\omega_{2A}, \text{ and}$$
$$p_1x_{1B}+p_2x_{2B} \leq p_1\omega_{1B}+p_2\omega_{2B}\}$$

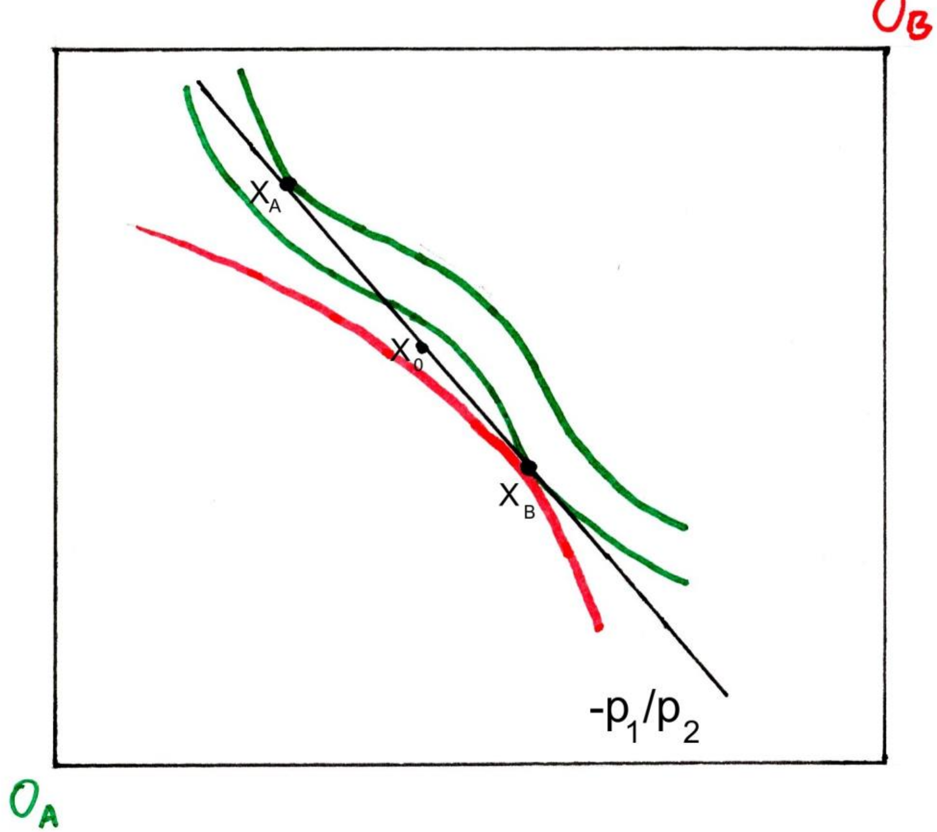
# Welfare economics theorems

Two fundamental theorems in welfare economics:

- They establish a relationship between Pareto optima (PO) and Walras equilibria (WE).
- The first: WE$\Rightarrow$PO,
- The second: PO$\Rightarrow$WE.
- Almost an equivalence.

# Welfare economics theorems

PO $\neq$ WE



## (T.30) The first fundamental welfare economics theorem

In an Edgeworth box pure exchange economy, if $X^*$ is a Walrasian equilibrium then $X^*$ is in the core of a corresponding coalitional game with transferable payments

Proof:

It is sufficient to demonstrate that no coalition can improve upon $X^*$. If exchanges were to be carried out at given prices this would have been obvious given both inequalities (no improvements are possible) and the feasibility condition (each consumer can spend not more than he/she obtains from selling his/her endowments). But the core property is more general. It refers to potential coalitions that do not necessarily apply price mechanisms.

Proof (continued):

Thus let us assume that there is another feasible allocation X' which is at least as good as $X^*$ for one consumer and strictly better for the other one. Then $p_1(x'_{1A}+x'_{1B})+p_2(x'_{2A}+x'_{2B})>p_1(x^*_{1A}+x^*_{1B})+p_2(x^*_{2A}+x^*_{2B})$, which means that X' must have been too expensive to be chosen in equilibrium. However, in a pure exchange economy: $x'_{1A}+x'_{1B}=x^*_{1A}+x^*_{1B}=\omega_1$, and $x'_{2A}+x'_{2B}=x^*_{2A}+x^*_{2B}=\omega_2$. Hence $p_1\omega_1+p_2\omega_2>p_1\omega_1+p_2\omega_2$ which is a contradiction.

(D.44)

A value function $\varphi_i$ defined for each player in a coalitional game with transferable payoffs <N,v> is called efficient if $\sum \varphi_i(N,v)=v(N)$; i.e. 'no value is wasted'. (Summation extends over all the players.)

(D.45)

A value function $\varphi_i$ defined for each player in a coalitional game with transferable payoffs <N,v> is called symmetric if $\varphi_i(N,v)=\varphi_h(N,v')$, where v and v' are identical except that the roles of players i and h are permuted.

<u>(D.46)</u>

A value function $\varphi_i$ defined for each player in coalitional games with transferable payoffs $<N,v_1>$ and $<N,v_2>$ is called <u>additive</u> if

$$\varphi_i(N,v_1+v_2)= \varphi_i(N,v_1)+\varphi_i(N,v_2),$$ where

$<N,v_1+v_2>$ is a coalitional game with transfers defined through $(v_1+v_2)(K)=v_1(K)+v_2(K)$

<u>(D.47)</u>

A value function $\varphi_i$ defined for each player in a coalitional game with transferable payoffs <N,v> satisfies the <u>dummy axiom</u> if $\varphi_i(N,v)=0$ for a player who does not contribute to any coalition, i.e. for every coalition K, $v(K \cup \{i\})=v(K)$.

<u>(T.31)</u>

A value function $\varphi_i$ defined for each player in a coalitional game with transferable payoffs <N,v> satisfying definitions D.44-D.47 is a Shapley value (difficult to prove)

# Questions:

Q-13 In a pure exchange economy, for any endowment the contract curve is a core of a corresponding game, because

[a] any other allocation gives them (jointly) less goods than it is feasible given their endowments

[b] <u>it is not possible to jointly improve their situation once they are in the contract curve</u>

[c] every point in the contract curve can be achieved given the initial allocation and the market prices

[d] the players agree to take into account not only their own respective utilities

[e] none of the above

# Exercises:

E-13 Calculate the Walrasian equilibrium (including the price ratio) in a pure exchange economy game where the two players have the allocations of the two goods (4,4) and (6,1), respectively, and their indifference curves are given by the following hyperboles: $x_{2A}=\alpha/x_{1A}$, and $x_{2B}=\beta/x_{1B}$, respectively ($\alpha,\beta>0$ – parameters).

# ***Coalitional games with transferable payments: Axiomatic bargaining***

## **Overall assumptions:**

1. The coalition consists of all (I) players
2. All players refer to the same 'starting point'
3. If no agreement is reached – the 'starting point' continues; thus it can be considered a 'threat'

## **Objective of the analysis:**

Find rules for players' desired behaviour

<u>(D.48)</u>

- U – the set of all possible payoff combinations,
- $U \subset \Re^{I}$
- U is closed and convex
- U satisfies <u>'free disposal' property</u> (i.e. if $\mathbf{u} \in U$, and $\mathbf{u}' \leq \mathbf{u}$ then $\mathbf{u}' \in U$)

Note:

Non-zero two-person games as defined in D.14 do not satisfy D.48, because their payoff sets are finite. However, later on, D.14 was extended to cover infinite payoff sets. As before, elements of U can be interpreted as (expected) utilities.

(D.49)

A function f: $2^{\Re^l} \rightarrow \Re^l$, a rule that assigns a solution vector $f(U) \in U$ to every bargaining problem, is called a bargaining solution

(D.50)

The bargaining solution f is independent of utility origins (IUO) if $f(U)=f(U')$, where

$$U'=\{\mathbf{u'} \in \Re^l: \exists \mathbf{u} \in U[\mathbf{u'}=(u_1+\alpha_1,\ldots,u_l+\alpha_l)]\}$$

for some $\alpha=(\alpha_1,\ldots,\alpha_l) \in \Re^l$. It is understood that in U' the 'starting point' $\mathbf{u'^*}=(u^*_1+\alpha_1,\ldots,u^*_l+\alpha_1)$ where $\mathbf{u^*}$ is the 'starting point' in U.

<u>(D.51)</u>

The bargaining solution f is <u>independent of utility units</u> (IUU), or <u>invariant to independent changes of units</u>, if for any $\mathbf{0}<\boldsymbol{\beta}\in\Re^I$, $f_i(U')=\beta_i f_i(U)$ for all $i=1,\ldots,I$, whenever $U'=\{(\beta_1 u_1,\ldots,\beta_I u_I)\colon \mathbf{u}\in U\}$

<u>(D.52)</u>

The bargaining solution f satisfies the <u>Pareto property</u> (P), or is <u>Paretian</u>, if for every U f(U) is a weak Pareto optimum, i.e. there is no $\mathbf{u}\in U$ such that $u_i>f_i(U)$ for every i.

<u>(D.53)</u>

The bargaining solution f is <u>symmetric</u> (S) if whenever U is a symmetric set (i.e. U remains unaltered under permutations of axes) all the coordinates of f(U) are equal (i.e. $f_1(U)=\ldots=f_I(U)$)

<u>(D.54)</u>

The bargaining solution f satisfies <u>individual rationality</u> (IR) whenever $f(U)\geq\mathbf{u}^*$

<u>(D.55)</u>

The bargaining solution f is <u>independent of irrelevant alternatives</u> (IIA) if whenever $U'\subset U$ and $f(U)\in U'$ it follows that $f(U')=f(U)$

## Note

For notational simplicity, it will be assumed from now on that $\mathbf{u}^*=\mathbf{0}$ (justified by D.50 and D.51)

## Example I (Egalitarian solution)

It is also called the Rawlsian solution: $f^e(U)= =(u^e,\ldots,u^e)$ where $u^e=\max\{\min\{u_1,\ldots,u_I\}: \mathbf{u}\in U\}$

## Example II (Utilitarian solution)

$f^u(U)=(u^u_1,\ldots,u^u_I)$ where $u^u_1+\ldots+u^u_I \geq u_1+\ldots+u_I$ for all $\mathbf{u}\in U\cup\Re_+^I$ ($f^u$ maximizes the sum of utilities)

## Example III (Nash solution)

$f^n(U)=(u^n{}_1,\ldots,u^n{}_l)$ where $u^n{}_1\times\ldots\times u^n{}_l \geq u_1\times\ldots\times u_l$ for all $\mathbf{u}\in U\cap\Re_+{}^l$ ($f^n$ maximizes the product of utilities). Alternatively it can be defined as a maximizer of the sum of logarithms of utilities $\mathit{ln}\,u_1+\ldots+\mathit{ln}\,u_l$.

## Example IV (Kalai-Smorodinski solution)

$f^k(U)=(u^k{}_1,\ldots,u^k{}_l)$ where $u^k{}_i=\delta u^{max}{}_i$ with

- $u^{max}{}_i=\max\{u_i: \mathbf{u}\in U\cap\Re_+{}^l\}$ and
- $\delta=\max\{\lambda: \lambda(u^{max}{}_1,\ldots,u^{max}{}_l)\in U\cap\Re_+{}^l\}$

# Example V (Nash vs. Kalai-Smorodinsky)

| | | Player 2 | |
|---|---|---|---|
| | | L | R |
| Player 1 | U | (0,3) | (0,0) |
| | D | (0,0) | (6,0) |

Two Nash equilibria: (U,L), and (D,R).

- Nash solution: split 6 equally: 3+3.
- Kalai-Smorodinsky solution: split 6=4+2.

$\delta$ in the example IV above is equal to 2/3.

(T.32)

The Nash solution is the only bargaining solution which satisfies IUO, IUU, P, S, and IIA

(difficult to prove)

(T.33)

The Nash solution satisfies IR

Proof:

By definition, $f^n(U) \geq \mathbf{0}$ as a product of non-negative numbers. By IUO and IUU $\mathbf{0}$ can be chosen as the 'starting point'.

# Note

High school graduates:

- A scored 20 in mathematics and 90 in history,
- B scored 50 in mathematics and 50 in history.

| | M | H | AA | GA |
|---|---|---|---|---|
| A | 20 | 90 | 55 | 42 |
| B | 50 | 50 | 50 | 50 |

- A better than B according to AA
- B better than A according to GA

# Questions:

Q-14 Utilitarian solution to a bargaining problem relies on the assumption that
[a] all players are characterized by identical utility functions
[b] players who (as a result of the game) are better off are given priority
[c] players who (as a result of the game) are worse off are given priority
[d] no players can improve their situation at the expense of others
<u>[e] none of the above</u>

# Exercises:

E-14 Calculate the egalitarian, utilitarian, Nash, and Kalai-Smorodinski solutions for the 'glove game' from E-12

# Evolutionary game theory

## Are Nash equilibria stable?

- Game structure
- Environment structure

Example I (Cournot duopoly)

- Two producers supply the same market
- The price is implied by the total supply:
  $p=a-b(y_1+y_2)$
- Both firms have identical cost functions and
  $MC_1=MC_2=AC_1=AC_2=c=const$
- The first rival makes a quantity decision $y_1$ that maximizes its profit expecting that the second rival does the same (with respect to $y_2$)
- Thus they solve two maximization problems:
  - $\max_{y1}\{(a-b(y_1+y_2))y_1-cy_1\}$
  - $\max_{y2}\{(a-b(y_1+y_2))y_2-cy_2\}$

# Example I (continued)

FOC for these problems are:

- $a-2by_1-by_2-c=0$
- $a-2by_2-by_1-c=0$

Solving these equations yields

- $y_1 = y_2 = (a-c)/3b$
- $y = 2(a-c)/3b$
- $p = (a+2c)/3$

This a 'standard' Cournot-Nash solution

# Example I (continued)

Rewriting the FOC one can yield

- $y_1=(a-c)/(2b)-y_2/2$, and
- $y_2=(a-c)/(2b)-y_1/2$

These are so-called reaction curves, i.e. the best response (profit maximizing output) of one player given the decision (output) of the other one. To simplify calculations from now on, we put specific numbers ($a=9$, $b=1$, and $c=3$). Reaction curves are thus $y_1=3-y_2/2$, and $y_2=3-y_1/2$, and the Nash equilibrium is (2,2).

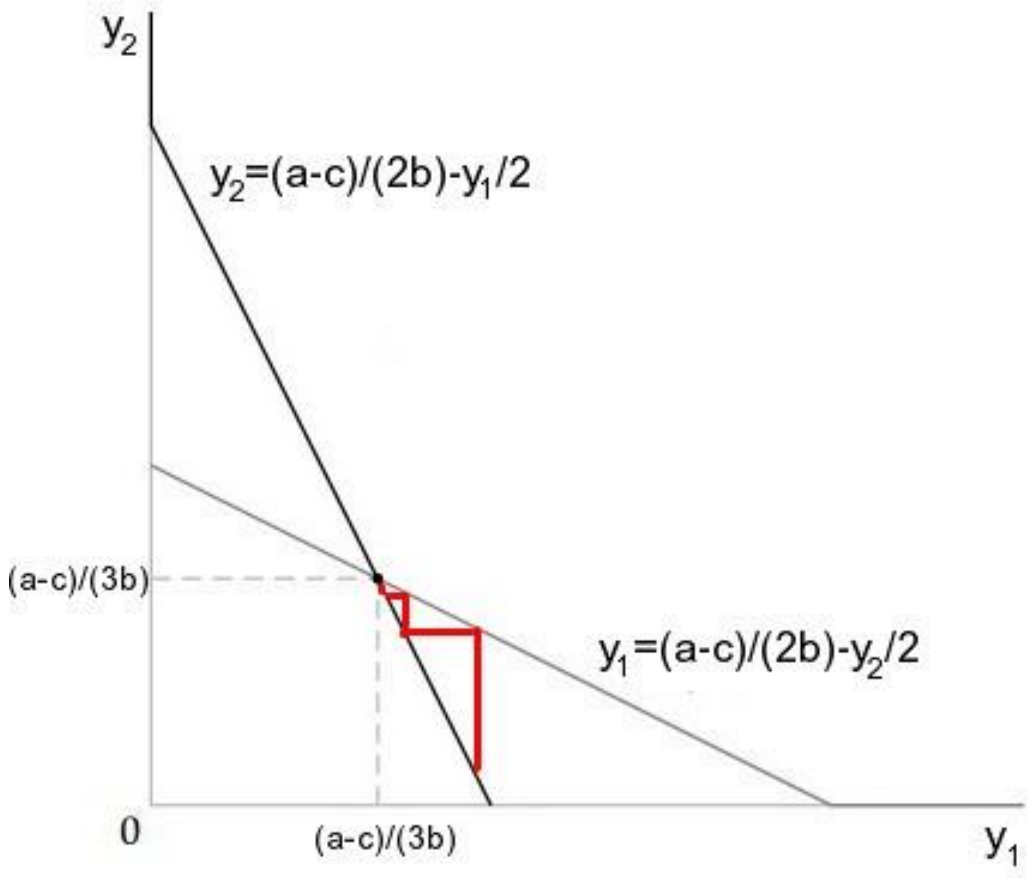
## Example I (continued)

Let us assume now that players missed the equilibrium. For instance, let $y_1^0=3$ (instead of 2). Assuming that this choice will be repeated, in the second round the rival may wish choose $y_2^1=3-3/2$ (according to the reaction curve), i.e. $y_2^1=3/2$. If the first player expects this choice to be repeated then his/her best reaction in the next round will be $y_1^2=3-(3/2)/2=9/4$. In yet another round the likely supply offered by the second one may be $y_2^3=3-(9/4)/2=15/8$. Then $y_1^4=3-(15/8)/2=33/16$, $y_2^5=3-(33/16)/2=63/32$, and so on.

## Example I (continued)

It is easy to see that both players are likely to approach (2,2), i.e. the Nash equilibrium. The example suggests that the Nash equilibrium in the Cournot game is stable. Even if the players miss it, they are likely to move towards it if they play the game repeatedly.

# Convergence in Cournot model



# Example II ('Cobweb model')

The so-called 'cobweb model' refers to a competitive market characterized by a demand curve $p=a-by$, and a supply curve $p=c+dy$ ($a,b,c,d>0$, and $a>c$). Equilibrium quantity is $y=(a-c)/(b+d)$, and equilibrium price is $p=(ad+bc)/(b+d)$. An additional assumption is that adjustments are done in discrete time intervals (e.g. years), suppliers react to market prices, but the quantity they offer in the next period is not flexible (i.e. it corresponds to the previous period price rather than to the present one).

## Example II (continued)

This can be modelled as a game with one player determining the demand and the other one – the supply. Assuming that disequilibrium is always considered as a bad outcome (if the price is higher than expected, suppliers regret that they did not offer more; if the price is lower than expected, suppliers regret that they did not offer less), its Nash equilibrium is when the actual price is equal to the expected one.

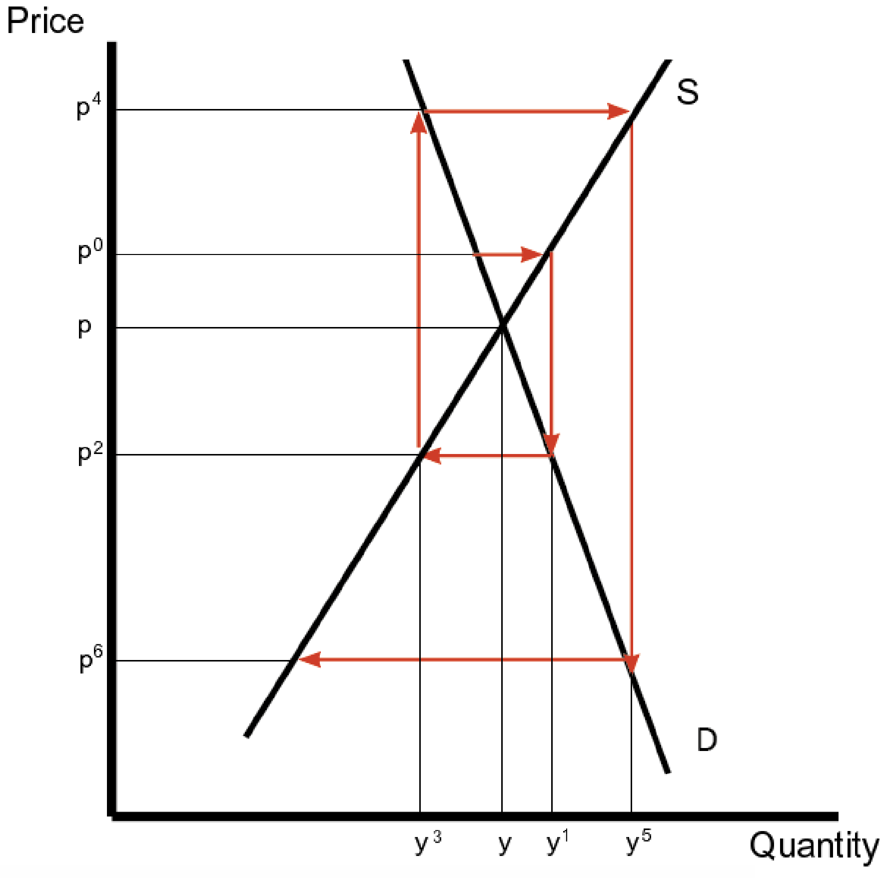
## Example II (continued)

If the equilibrium quantity and price were achieved, then $y^t=(a-c)/(b+d)$ and $p^t=(ad+bc)(b+d)$ for $t=0,1,2,....$ Let us see what happens if the price faced by the suppliers was not their expected price. To simplify calculations from now on, we put specific numbers ($a=12$, $b=2$, $c=0$, $d=1$). Thus the demand curve is $p=12-2y$, and the supply curve is $p=y$. The equilibrium quantity is $y=12/3=4$, and the equilibrium price is $p=12/3=4$.

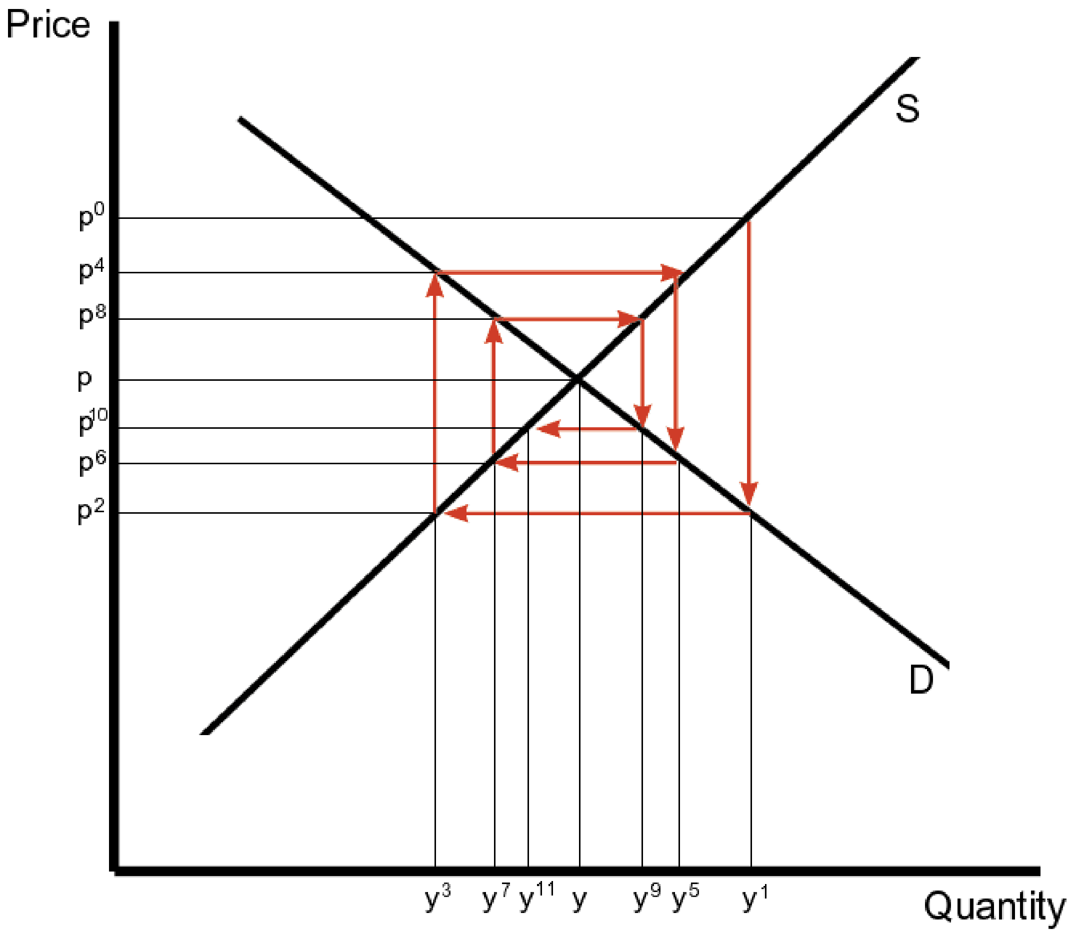
## Example II (continued)

Let us assume that for whatever reason $p^0=5$. Reacting to this price the supply will be $y^1=5$. With this quantity offered, the demand will bring the price to $p^1=2$. The next period supply will be $y^2=2$, and the price $p^2=8$. It is easy to see that the quantities and prices will diverge from the equilibrium. Hence the equilibrium outcome is not stable: if players miss the equilibrium they will never return to it.

# Cobweb model (divergence)



# Cobweb model (convergence)



# Example III (Mutation)

Let us consider the following 'Dove-Hawk' game

| | | Second | |
|---|---|---|---|
| | | D | H |
| First | D | (4,4) | (0, 8) |
| | H | (8,0) | (**3,3**) |

## Example III (continued)

If two animals find a prey they can behave like doves, i.e. share it peacefully. If one behaves like a dove and the other like a hawk, the militant one gets everything and the other one is left with nothing. If both of them behave like hawks, they loose a quarter of the prey and share the remaining part equally. The game has a unique Nash equilibrium (H,H). In other words, all animals are likely to behave as hawks.

## Example III (continued)

Now let us assume that in principle all animals behave like hawks, but there are few 'mutants' who behave like doves. Their behaviour is hereditary, but an individual never changes it. Thus the 'game' they play has a probabilistic structure. The payoffs depend on probabilities of encounters (D-D), (D-H or H-D; the matrix is symmetric), and (H-H).

## Example III (continued)

Let us assume that ε is the proportion of 'mutants' (i.e. those who turn out to be doves). Thus the proportion of 'non-mutants' is 1-ε. The expected outcome of the encounter of two individuals (in terms of 'payoffs' of the Dove-Hawk game) is 4ε for the 'mutant' and 8ε+3(1-ε) for a 'non-mutant'. The second number is larger than the first one, so the 'mutant' population is likely to have smaller off-spring and consequently its proportion will go down to zero.

## (D.56) Replicator game

Let us assume that a population consists of two types of individuals who play a symmetric D.14 game. However, they do not choose strategies, but each is determined to apply a certain pure strategy and they 'pass it over' to their offspring. Let $\varepsilon_i$ be the statistical frequency of individuals who play strategy i ($\varepsilon_1+\varepsilon_2=1$). It is further assumed that the population reproduces asexually (i.e. an individual has offspring without matching) and 'reproduction success' is proportional to the payoff from the game.

# <u>(D.57) Evolutionary Stable Strategy</u>

A population plays a Replicator game (D.56). Let there be a pure-strategy Nash equilibrium $(i^*, i^*)$ in the game. It is called an <u>Evolutionary Stable Strategy (ESS)</u> if

$$\varepsilon_{i^*} P_{i^*,i^*} + \varepsilon_{-i^*} P_{i^*,-i^*} > \varepsilon_{-i^*} P_{-i^*,i^*} + \varepsilon_{i^*} P_{i^*,-i^*}.$$

<u>Note</u>

Being 'hawkish' in the 'Dove-Hawk' game is an ESS

<u>Note</u>

Definition of ESS can be extended to more complicated games, e.g. consisting of more than two strategies

# Note

There are Nash equilibria that do not determine an ESS. To see this, consider the game

| | | Second | |
|---|---|---|---|
| | | I | II |
| First | I | (**1,1**) | (0,0) |
| | II | (0,0) | (**0,0**) |

It has two Nash equilibria – (1,1) and (0,0) – but only the first one determines an ESS

# (D.58) Replicator equation

A population plays a Replicator game. Let $\varepsilon_i(t)$ be the frequency of individuals of type i in the population at time t. A formula determining $\varepsilon_i(t)$ as a function (perhaps an indirect one) of time and the game characteristics is called a Replicator equation.

# Example IV (Replicator equation)

The formula from D.58 may take the following (differential) form:

$$d\varepsilon_i(t)/dt = \varepsilon_i(t)(f_i(\varepsilon_i(t))-\varphi(\varepsilon_i(t))), \text{ where}$$

$$\varphi(\varepsilon_i(t))=\varepsilon_1 f_1(\varepsilon_i(t))+\varepsilon_2\, f_2(\varepsilon_i(t)).$$

$f_i$: $[0,1]\rightarrow\Re$ is a 'fitness' of type i to the environment characterized by the game and distribution of types in the population, and $\varphi$ is the average 'fitness' in the population.

# Example IV (continued)

The Replicator equation predicts that the share of the type that is less fit than the average will decrease, and the share of the type that is more fit than the average will increase. It also lets estimate how fast the process will be.

# Note

The 'Dove-Hawk' game (Example III) predicts that the share of 'dovish' individuals will shrink, but – without a Replicator equation – it cannot be more specific about the pace of this process.

# Note

The definition of Replicator game (D.56) can be modified by assuming that the population is homogeneous (one type only), but there are two strategies to choose. As before, $\varepsilon_i$ is the statistical frequency of individuals who play strategy i ($\varepsilon_1+\varepsilon_2=1$). Unlike in the original definition, playing a strategy is not only inherited, but it can also be changed by an individual. The change of $\varepsilon_i$ is determined by a modified 'Replicator equation': $d\varepsilon_i(t)/dt = \alpha\beta\varepsilon_i(t)(f_i(\varepsilon_i(t))-\varphi(\varepsilon_i(t)))$, where $\alpha$ – learning rate (how fast individuals learn that changing strategy can be beneficial), and $\beta$ – willingness to change (if an individual knows that the other strategy is more attractive, is it willing to change the behaviour); other symbols are as in D.58

Note

D.57 (ESS) can be extended to account for Replicator games with more complicated characteristics (such as learning/adoption).

(T.34)

Let $(i^*,i^*)$ be an ESS. Then the system of Replicator equations $d\varepsilon_i(t)/dt = \varepsilon_i(t)(f_i(\varepsilon_i(t))-\varphi(\varepsilon_i(t)))$ for $i=1,2$ has a stationary solution, $\varepsilon_{i^*}(t)=1=\text{const.}$

(difficult to prove)

# Questions:

Q-15 An Evolutionary Stable Strategy (ESS) in a population where a mutation appeared
[a] assumes that mutants do not have offspring
<u>[b] assumes that individuals using ESS reproduce more successfully than mutants</u>
[c] cannot be reconciled with asexual reproduction
[d] leads to gradual elimination of individuals who fail to adopt new reproduction techniques
[e] none of the above

# Exercises:

E-15 The 'Dove-Hawk' game from Example III is supplemented with the following 'Replicator equation': if the expected payoff of the mutant in the game at time t is lower than 1 then its share in the population halves every period $\varepsilon(t+1)=\varepsilon(t)/2$; if the expected payoff of the mutant in the game at time t is lower than 1/10 then $\varepsilon(t+1)=0$. Please discuss how much time it will take to eliminate mutants from the population.

# Outline solutions to exercises

E-1. The 'only if' part is trivial, so one just needs to prove the 'if' part. The proof boils down to observing that any number $\alpha \in [0,1]$ can be approximated by a series $\alpha_1/2^1+\alpha_2/2^2+\alpha_3/2^3+...$ (it can be approximated by a binary number; numbers $\alpha_1, \alpha_2, \alpha_3, ... \in \{0,1\}$ are binary digits of this approximation). However, completing the proof requires the continuity assumption; which is not included in the original definition (D.3), but added in some theorems (as in T.4)

E-2. A constant Arrow-Pratt coefficient of absolute risk aversion A>0 means that A=-u''(x)/u'(x). This is equivalent to u''(x)=-Au'(x). But u''(x)=(u'(x))'. Thus -Au'(x)=(u'(x))'. By integrating both sides (twice – and hence two constants may show up) we get $u(x)=-e^{const}\cdot e^{-Ax}+const$.

E-3. Let us take two lotteries with identical means: $L=(p_1,...,p_N)$ ,$L'=(p'_1,...,p'_N)$ with $p_1x_1+...+p_Nx_N=p'_1x_1+...+p'_Nx_N$. To prove the first implication, we assume that $p_1u(x_1)+...+p_Nu(x_N) \geq p'_1u(x_1)+...+p'_Nu(x_N)$ for every nondecreasing function $u:X\rightarrow\Re$. The second order stochastic dominance requires that the inequality holds for nondecreasing concave functions u and therefore it is obviously satisfied. To see that the converse is not true, one needs to observe that holding the inequality for concave functions does not imply that it also holds for non concave functions.

E-4. The strategy U of the first player is understood as choosing the same location as the strategy L of the second player. Likewise, the strategy D of the first player is understood as choosing the same location as the strategy R of the second player. The players derive high utility if they meet and low utility if they do not meet (but they cannot communicate). The game has two Nash equilibria with payoffs (1,1).

E-5. Achieving (4,4) implies that the outcome (U,R) is ruled out for sure. If, however, the first player knows that the second chooses L, then he/she would choose U rather than D. And *vice versa*, if the second knows that the first chooses D then he/she would choose R rather than L. Thus (U,R) cannot be ruled out for sure. If players chose their strategies randomly, each would achieve the payoff of 2.5. They may coordinate their choices by a signalling device which has three states (A, B, or C), each with the probability of 1/3 that are not perfectly recognized by the players. If A occurs the player 1 knows this, but if the state is B or C, the player does not know whether it is B or C. Conversely, the player 2 is perfectly informed about C, but he/she cannot distinguish between A and B. The correlated Nash equilibrium is: U when told A, and D when told (B,C); R when told C, and L when told (A,B). Please check that neither of the players has incentives to unilaterally deviate from these strategies, and – on average – they yield payoffs 3 1/3 for each of them. Please also note that (U,L), (D,L), and (D,R) are used with 1/3 probabilities each, so that the worst outcome – (U,R) – is avoided.

E-6. Replicating the calculations from the class example one gets the probability $\mu=1/4$. Thus the Bayesian Nash equilibrium is a pair of strategies $(S_P(\theta_P),S_D(\theta_D))$ where $S_P(\theta_P)=S_P=($C if $\mu>1/4$, D if $\mu<1/4)$, and $S_D(\theta_D)=($C if the realization of $\theta_D$ is type "I", D if the realization of $\theta_D$ is type "II").

E-7. The craftsman's optimization problem reads: $\text{Max}_L\{40L^{1/2}-0.5L-R\}\geq 300$ (the inequality is the participation constraint). To solve the maximization problem we disregard R (since it does not depend on L) and differentiate the expression $40L^{1/2}-0.5L$. The derivative reads $20L^{-1/2}-0.5$ (equated to zero, this is the incentive compatibility constraint). It vanishes when L=1600. With such a labour input, the craftsman's revenue is 800-R. Thus R≤500 if the contract is to be accepted.

E-8. Zermelo theorem may still hold if in terminal nodes $x_1$ and $x_2$ there are identical payoffs for the player who does not move in the node $p(x_1)=p(x_2)$. When this happens, but the player who moves in $p(x_1)=p(x_2)$ has different payoffs in various nodes of $s(p(x_i))$, the action to be taken in $p(x_i)$ can be determined unambiguously.

E-9. In the beginning only I supplies the market as a monopolist. Thus its maximum profit is $\pi^m= (a-c)^2/(4b)$. If F decides to enter the rivals decide whether to cooperate or to attack. Cooperation means setting own supply at the Cournot level, i.e. $q_i=(a-c)/(3b)$, while an attack – setting own supply at the monopolistic level (as if the firm was the only supplier), i.e. $q_i=(a-c)/(2b)$. The profits result from the total supply. If both cooperate, they make the Cournot profit, i.e. $\pi^c= (a-c)^2/(9b)$ each. If one sets the supply at the monopolistic level and the other at the Cournot one, then the total supply is 5(a-c)/(6b). The market will establish the price (a+5c)/6. The one who wanted to cooperate will make the profit $(a-c)^2/(18b)$, and the one who wanted to attack:

| | | Incumbent (I) | |
|---|---|---|---|
| | | C if F enters | A if F enters |
| Entering Firm (F) | D&C | $0,(a-c)^2/(4b)$ | $0,(a-c)^2/(4b)$ |
| | D&A | $0,(a-c)^2/(4b)$ | $0,(a-c)^2/(4b)$ |
| | E&C | $(a-c)^2/(9b),(a-c)^2/(9b)$ | $(a-c)^2/(18b),(a-c)^2/(12b)$ |
| | E&A | $(a-c)^2/(12b),(a-c)^2/(18b)$ | 0,0 |

$(a\text{-}c)^2/(12b)$. If both attack the total supply will be (a-c)/b, the market will establish the price c, i.e. wiping out all the profits. Thus the payoff matrix reads as above. If b>0 then the matrix is

| | | Incumbent (I) | |
|---|---|---|---|
| | | C if F enters | A if F enters |
| Entering Firm (F) | D&C | 0,α | 0,α |
| | D&A | 0,α | 0,α |
| | E&C | β,β | δ,γ |
| | E&A | γ,δ | 0,0 |

with α>β>γ>δ>0. Thus SPNE is (β,β) (i.e. (E&C,C) as in the original game analyzed in the class), but – unlike that from the class – this game does not have other Nash equilibria.

E-10. Of course, every game actually played has to be a finite one. Nevertheless, the players may not know how many rounds it has. Thus in any single round they do not have to be compelled (by the Folk theorem logic) to behave non-cooperatively. However, given the fact that any experiment lasts, say, no more than two hours (this information is commonly known), the players may contemplate probabilities that the round played is the last one. These probabilities increase in time, and they can approach 1 if the players sit in the lab almost two hours. Hence, the longer they play, the larger their incentive to expect that the round played is the last one.

E-11. In the common resource game defined in the class, players are likely to drive the resource to extinction. To see this, please note that $a^t_1=a^t_2=\Delta K^t/2$ is not a Nash equilibrium; if either of the players extracts more, then he/she will be better off than by extracting sustainably.

E-12. It may result in some changes, but players are likely to seek some other solutions. Faced with the risk of being excluded from the coalition, the third player is likely to offer to reduce his/her payoff, perhaps to the symmetric profile of [1/3,1/3,1/3], but the first player may then suggest to get rid of the second one, and to get $\alpha=4/5$ to be split into halves (2/5 each). This will improve the hypothetical situation of the first and the third at the expense of the second one. No matter how they manipulate with payoffs, there will always be somebody who can be made better off by suggesting an alternative coalition. Hence as long as $\alpha=4/5$, the core of the game is empty.

E-13. The answer is [6,3,4,2] and $p_1=p_2/2$. At the first glance it seems as if we had 7 unknowns: $x_{1A}$, $x_{2A}$, $x_{1B}$, $x_{2B}$, $\alpha$, $\beta$, and $p_1/p_2$. Nevertheless this number can easily be reduced to 5, since $x_{1B}=10- x_{1A}$ and $x_{2B}=5- x_{2A}$. Thus let $p=p_1/p_2$, $x_1=x_{1A}$ and $x_2=x_{2A}$. By the definition of the indifference curves: $\alpha=x_1x_2$, and $\beta=(10-x_1)(5-x_2)$. In the Walrasian equilibrium both indifference curves are tangent to the price ratio, i.e. $-\alpha/(x_1)^2=-p$, and $-\beta/(10-x_1)^2=-p$, which lets calculate $\alpha$ and $\beta$: $\alpha=p(x_1)^2$ and $\beta=p(10-x_1)^2$. Thus from the definitions of the indifference curves: $p(x_1)^2=x_1x_2$ and $p(10-x_1)^2=(10-x_1)(5-x_2)$; and after cancellations we get: $px_1=x_2$ and $p(10-x_1)=(5-x_2)$. These are 2 equations with 3 unknowns. We need an additional one. By the necessity of equating expenditures with revenues (please note that the same equation is derived irrespective of whether the balance reflects the consumer A or B) we get $p_1(4-x_1)=p_2(x_2-4)$, i.e. $p_1/p_2=(4-x_2)/(x_1-4)$ (with $x_1\neq4$). Hence the third equation reads $p=(4-x_2)/(x_1-4)$. By substituting p into the previous equations we get: $x_1(4-x_2)=x_2(x_1-4)$ and $(10-x_1)(4-x_2)=(5-x_2)(x_1-4)$. These are two equations with two unknowns: $4x_1-2x_1x_2=-4x_2$ and $60-9x_1+2x_1x_2=14x_2$. By adding the two we get $x_2=6-x_1/2$ which can be substituted into the first one, yielding $(x_1)^2-10x_1+24=0$. The last quadratic equation has two roots: $x_1=6$ or $x_1=4$ implying $x_2=3$ or $x_2=4$, respectively. The second solution has to be rejected (in order to calculate p properly; i.e. – as noted above – to avoid dividing into zero).

E-14. The 'glove game' means producing a fully matched pair of gloves which has the value of one. The solutions mean how to distribute this value among its participants, i.e. players 1, 2, and 3. All but the utilitarian solutions are $u_1=u_2=u_3=1/3$. The utilitarian solution is not unique; any combination of positive numbers, as long as $u_1+u_2+u_3=1$ (for instance a combination $u_1=1$, $u_2=u_3=0$, or combination of Shapley values $u_1=u_2=1/6$, $u_3=2/3$) works.

E-15. First of all, it needs to be clarified that the 'Dove-Hawk' game is understood as an 'evolutionary' game with population of individuals who were aggressive (they behaved like hawks). All of a sudden there appeared a 'mutation' causing some individuals to be peacefully-minded (they behave like doves). We look at doves as 'mutants', whose share in the population is small ($\varepsilon(0)$ is non-negative but small). In order to see what will happen with the mutant offspring in period t=1 (to calculate $\varepsilon(1)$), we need to calculate the expected payoff for a mutant. The payoff enjoyed by such a mutant depends on who will be encountered: another mutant (a 'dovish' individual) or a non-mutant (a 'hawkish' individual). In the first case the payoff is 4, while in the second case it is 0. Assuming that encounters are random, then the first case happens with the probability of $\varepsilon(0)$, and the second – with the probability of $1-\varepsilon(0)$. Therefore the expected payoff is $4\varepsilon(0)$.

Thus if $1/10\leq 4\varepsilon(0)<1$, i.e. $0.025\leq\varepsilon(0)<0.25$ – then it may take 2,3,4, or 5 periods for $\varepsilon(t)$ to fall below 0.025 and consequently the expected payoff to fall below 1/10. If the expected payoff $4\varepsilon(0)\geq 1$ (i.e. $\varepsilon(0)\geq 0.25$) then the exercise does not specify what may happen. By definition, $\varepsilon(0)\leq 1$. Therefore we may be curious about what happens if this initial share ranges from 25%

to 100%. However in the exercise it is not specified what happens then. If we contemplate an initial share higher than 25% then the expected payoff will be higher than 1 which does not imply extinction in the exercise. But on the other hand, we interpret $\varepsilon(0)$ as a 'small' number, so for all practical purposes this information is not relevant.

To sum up, the 'dovish' mutation is not likely to survive (this is seen when we look at the replicator game). It will disappear immediately, or after a couple of years, depending on its initial share (this can be calculated using the replicator equation). If the initial share is close to 25%, it will survive up to the fifth period.